On the Vlasov Inverse problem and the Continuum Hamiltonian Hopf Bifurcation

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Goal: Describe the integral transform that diagonalizes continuous spectrum, and some of its uses: Solve Landau problem. Define signature for continuous spectrum. Motivates a Krein-like theorem for instabilities that emerge from the continuous spectrum. Motivates a nonlinear normal form (pde).

Overview

- Integral Transform and Inverse Problem
- Energy and Signature
- Continuum Hamiltonian Hopf Bifurcation
- Nonlinear Normal Forms – Single Wave Model, Hickernell-Berk-Breizman equation, ...

- **Integral Transform and Inverse Problem**

Integral Transforms

Fourier:

$$f(x) \longleftrightarrow g(k)$$

$$f(x) = \int_{\mathbb{R}} K(x,k) g(k)\, dk \qquad \text{where} \qquad K(x,k) = e^{ik\cdot x}$$

Hilbert:

$$f(v) \longleftrightarrow g(u)$$

$$f(v) = \int_{\mathbb{R}} K(u,v) g(u)\, du \qquad \text{where} \qquad K(u,v) = \frac{\mathbb{P}}{\pi} \frac{1}{u-v}$$

A General Transform for Continuous Spectra

Definition:

$$f(v) = G[g](v) \; := A(v)\, g(v) + B(v)\, H[g](v) = \int_{\mathbb{R}} K(u,v) g(u)\, du$$

where $A(v)$ and $B(v)$ are real valued functions of a real variable (K generalized function) such that:

$$B(v) = 1 + H[A](v)\,,$$

and the Hilbert transform

$$H[g](v) := \frac{1}{\pi} \fint \frac{g(u)}{u - v}\, du\,,$$

with $\fint$ denoting Cauchy principal value of $\int_{\mathbb{R}}$.

Actually subset of more general transform!

Transform Theorems

Theorem (G1). *$G\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:*

$$\|G[g]\|_p \leq C_p \, \|g\|_p \,,$$

where C_p depends only on p.

Theorem (G2). *If A is a good function, then $G[g]$ has an inverse,*

$$G^{-1}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= G^{-1}[f](u) \\ &:= \frac{B(u)}{A^2 + B^2} \, f(u) - \frac{A(u)}{A^2 + B^2} \, H[f](u) \,, \end{aligned}$$

Vlasov-Poisson System

Phase space density (1 + 1 + 1 field theory):

$$f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^+ , \qquad f(x, v, t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{e}{m} \frac{\partial \phi[x, t; f]}{\partial x} \frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = 4\pi \left[e \int_{\mathbb{R}} f(x, v, t)\, dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} v^2 f \, dx dv + \frac{1}{8\pi} \int_{\mathbb{T}} (\phi_x)^2 \, dx$$

Linear Vlasov-Poisson System

Expand about Stable Homogeneous Equilibrium:

$$f = f_0(v) + \delta f(x, v, t)$$

Linearized EOM:

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{e}{m}\frac{\partial \delta \phi[x,t;\delta f]}{\partial x}\frac{\partial f_0}{\partial v} = 0$$

$$\delta \phi_{xx} = 4\pi e \int_{\mathbb{R}} \delta f(x,v,t)\, dv$$

Linearized Energy (Kruskal-Oberman 1958):

$$H_L = -\frac{m}{2}\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{v\,(\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi}\int_{\mathbb{T}} (\delta\phi_x)^2\, dx$$

Landau's Problem

Assume

$$\delta f = \sum_k f_k(v,t)e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t)e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikvf_k + ik\phi_k \frac{e}{m}\frac{\partial f_0}{\partial v} = 0\,, \qquad k^2\phi_k = -4\pi e \int_{\mathbb{R}} f_k(v,t)\,dv$$

Three methods:

1. Laplace Transforms (Landau 1946)
2. Normal Modes (van Kampen 1955)
3. Coordinate Change $\Longleftrightarrow$ Integral Transform (PJM, Pfirsch, Shadwick, ... 1992, 2000, ...)

Transform Choice and Identities

Tailor Transform as follows:

$$A(v) := \epsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v} \quad \Rightarrow \quad B(v) := \epsilon_R(v) = 1 + H[\epsilon_I](v)\,,$$

General identities written out for this case

- G^{-1} is the inverse of G
- $$G^{-1}[vf](u) = u\, G^{-1}[f](u) - \frac{\epsilon_I}{|\epsilon|^2} \frac{1}{\pi} \int_{\mathbb{R}} f\, dv$$
- $$G^{-1}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$$

where $|\epsilon|^2 = \epsilon_I^2 + \epsilon_R^2$ and recall $\epsilon_I \sim f_0'$.

Inverse Proof

That G^{-1} is the inverse follows upon calculating $g = G^{-1}[G[g]]$, and using Hilbert transform identities:

$$\begin{aligned}
g(u) &= \widehat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u) \\
&= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, [\epsilon_R(u)\, g(u) + \epsilon_I(u)\, H[g](u)] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H\left[\epsilon_R(u')\, g(u') + \epsilon_I(u')\, H[g](u')\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H\left[H[\epsilon_I]\, g + \epsilon_I\, H[g]\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, [H[\epsilon_I](u) H[g](u) - g(u)\, \epsilon_I(u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\, [1 + H[\epsilon_I](u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\epsilon_R(u) = g(u)
\end{aligned}$$

$\square$

Solution

Solve like Fourier transforms: operate on EOM with $G^{-1} \Rightarrow$,

$$\frac{\partial g_k}{\partial t} + iku\, g_k - ik\frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{\mathbb{R}} f\, dv + ik\frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{\mathbb{R}} f\, dv = 0$$

$$\frac{\partial g_k}{\partial t} + iku\, g_k = 0$$

and so

$$g_k(u,t) = \mathring{g}_k(u)e^{-ikut}\,.$$

Using $\mathring{g}_k = G^{-1}[\mathring{f}_k]$ we obtain the solution

$$\begin{aligned} f_k(v,t) &= G[g_k(u,t)] \\ &= G\left[\mathring{g}_k(u)e^{-ikut}\right] = G\left[G^{-1}[\mathring{f}_k]e^{-ikut}\right] \end{aligned}$$

Equivalant to van Kampen's and Landau's solution!

Inverse Problem?

What is $g_k(u,t)$ physically? The van Kampen mode electric field!

Sum over modes

$$E_k(t) = \int_{\mathbb{R}} \mathring{g}_k(u) e^{-ikut} du = \int_{\mathbb{R}} E_k(\omega) e^{-i\omega t} d\omega$$

where $E_k(\omega) = \mathring{g}_k(u)/|k|$.

Usual Logic: Choose $\mathring{f}_k \to \mathring{g}_k$ such that

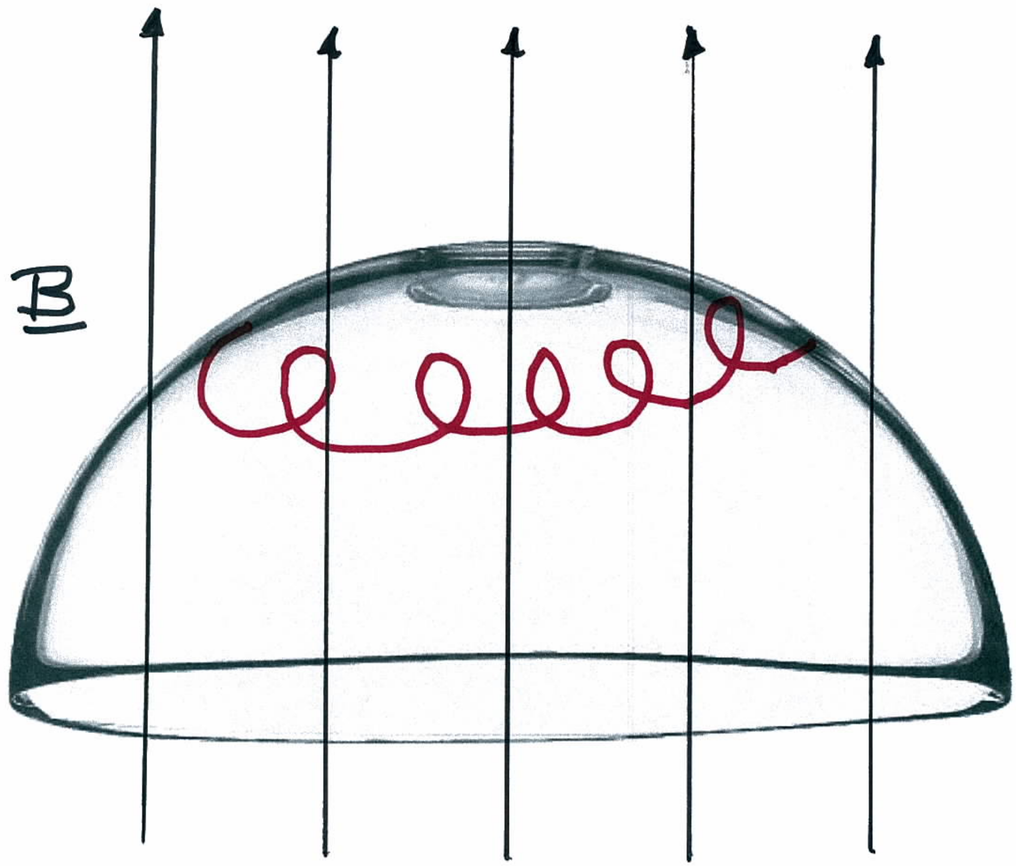
$$\lim_{t\to\infty} E_k(t) \sim e^{-\gamma_L t}$$

Why? Riemann-Lebesgue Lemma: γ determined by closest pole to real axis when $\mathring{g}_k(u)$ continued into complex u-plane.

Inverse Logic: Choose $E_k(t) \to \mathring{g}_k \to \mathring{f}_k$. Note, $E_k(t)$ can have ANY $t \to \infty$ asymptotic behavior. Price paid is strange $\mathring{f}_k$. (special case due to Weitzner 1960s).

- **Energy and Signature**

Charged Particle on Slick Mountain



Falls and Rotates $\Rightarrow$ Precession

Realized in a uniformly charged column.

Charged Particle on Quadratic Mountain

Simple model of FLR stabilization $\rightarrow$ plasma mirror machine.

Lagrangian:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{eB}{2}\left(\dot{y}x - \dot{x}y\right) + \frac{K}{2}\left(x^2 + y^2\right)$$

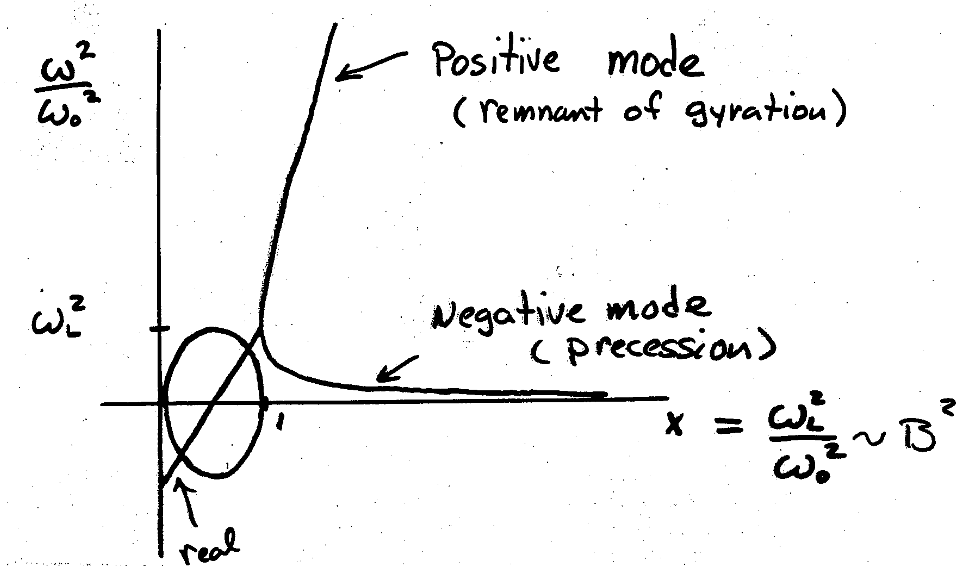
Hamiltonian:

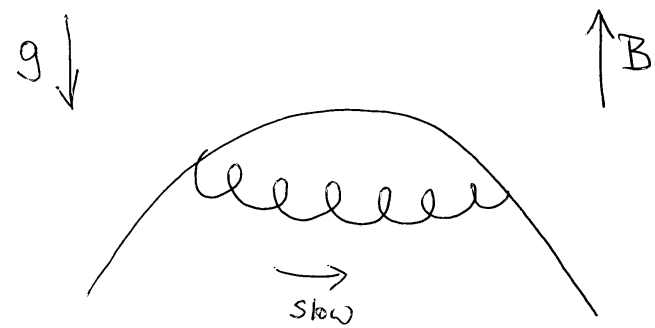
$$H = \frac{m}{2}\left(p_x^2 + p_y^2\right) + \omega_L\left(yp_x - xp_y\right) - \frac{m}{2}\left(\omega_L^2 - \omega_0^2\right)\left(x^2 + y^2\right)$$

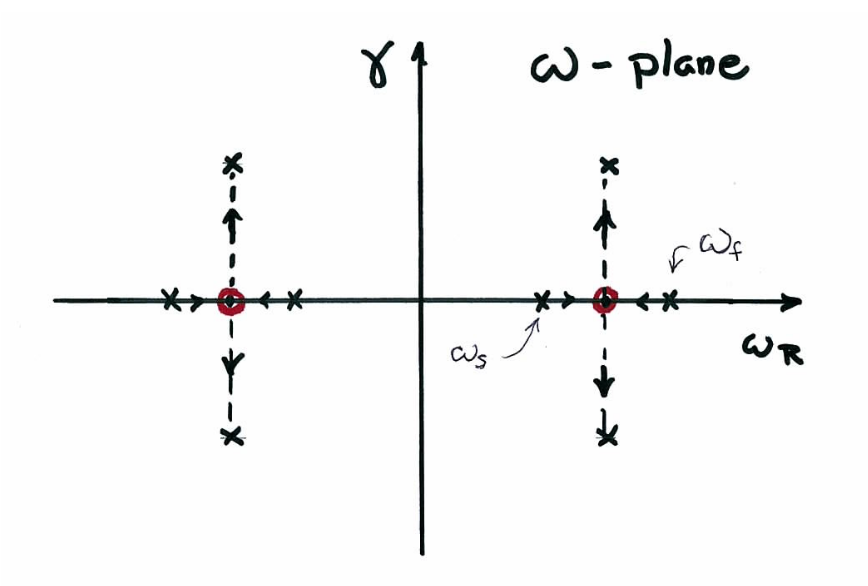
Two frequencies:

$$\omega_L = \frac{eB}{2m} \qquad \text{and} \qquad \omega_0 = \sqrt{\frac{K}{m}}$$

Hamiltonian Hopf Bifurcation (Krein Crash)







$$x, y \sim e^{i\omega t} = e^{\lambda t}$$

Quadratic Mountain Stable Normal Form

For large enough B system is stable and $\exists$ a coordinate change, a canonical transformation $(q,p) \to (Q,P)$, to

$$H = \frac{|\omega_f|}{2}\left(P_f^2 + Q_f^2\right) - \frac{|\omega_s|}{2}\left(P_s^2 + Q_s^2\right)$$

Slow mode is a negative energy mode – a stable oscillation that lowers the energy relative to the equilibrium state.

Weierstrass (1894), Williamson (1936), ...

.

- Hamiltonian normal form theory.

Krein-Moser Bifurcation Theorem

Krein (1950) – Moser (1958) – Sturrock (1958)

Such bifurcation to instability (with quartets) can only happen if colliding eigenvalues have opposite signature $\sigma_i \in \{\pm\}$, where

$$H = \sum_i \sigma_i |\omega_i| (p_i^2 + q_i^2)/2 = \sum_i \sigma_i |\omega_i| J_i$$

One must be a negative energy mode.

Sturrock looked at two-stream instability.

Vlasov in Class of Hamiltonian Field Theories

- plasma physics (charged particles-electrostatic)

- vortex dynamics, QG, shear flow

- stellar dynamics

- statistical physics (XY-interaction)

- ...

- general transport via mean field theory

Hamiltonian Structure

Noncanonical Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} dqdp\, f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] = \int_{\mathcal{Z}} dqdp\, F_f \mathcal{J} G_f = \left\langle f, [F_f, G_f]\right\rangle$$

Cosymplectic Operator:

$$\mathcal{J}\ \cdot = \frac{\partial f}{\partial p}\frac{\partial\ \cdot}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial\ \cdot}{\partial p}$$

Vlasov:

$$\frac{\partial f}{\partial t} = \{f, H\} = \mathcal{J}\frac{\delta H}{\delta f} = -[f, \mathcal{E}].$$

Casimir Degeneracy:

$$\{C, F\} = 0 \qquad \forall F \qquad \text{for} \qquad C[f] = \int_{\mathcal{Z}} dqdp\, \mathcal{C}(f)$$

Too many variables and not canonical.

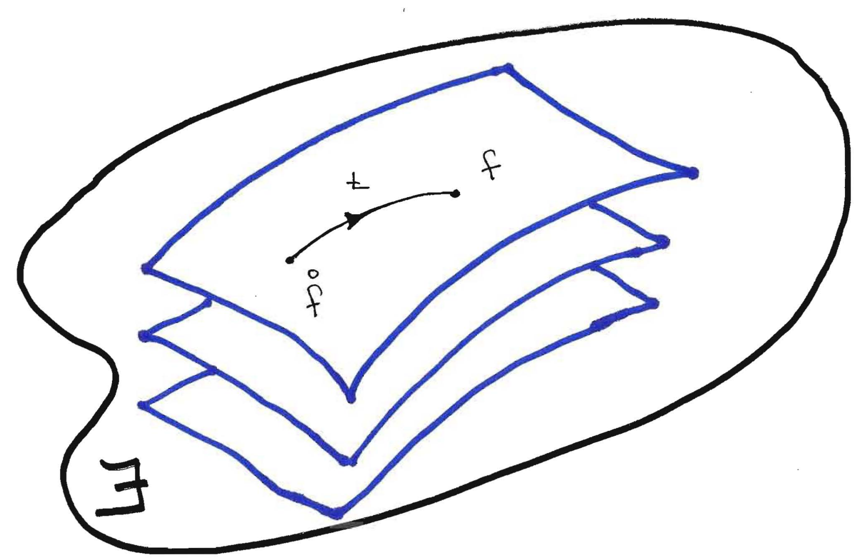
See Cartoon – Hamiltonian on leaf.

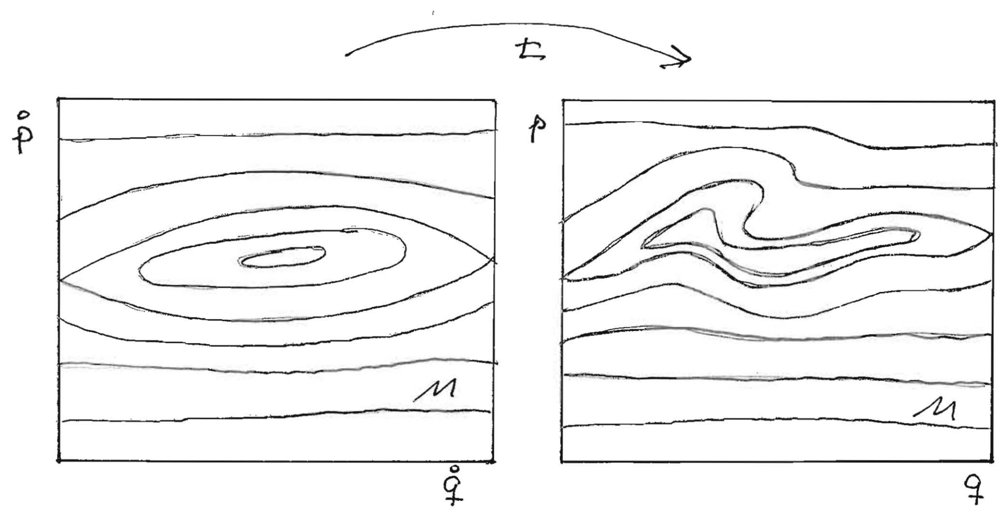
VP Cartoon– Symplectic Rearrangement

$f(x,v,t) = \mathring{f} \circ \mathring{z}$

$f \sim g$ if $f = g \circ z$

with z symplectomorphism





$p = mv$

μ volume measure

$f(x,v,t) = \mathring{f}(\mathring{x}(x,v,t), \mathring{v}(x,v,t))$

Linear Hamiltonian Theory

Expand f-dependent Poisson bracket and Hamiltonian $\Rightarrow$

Linear Poisson Bracket:

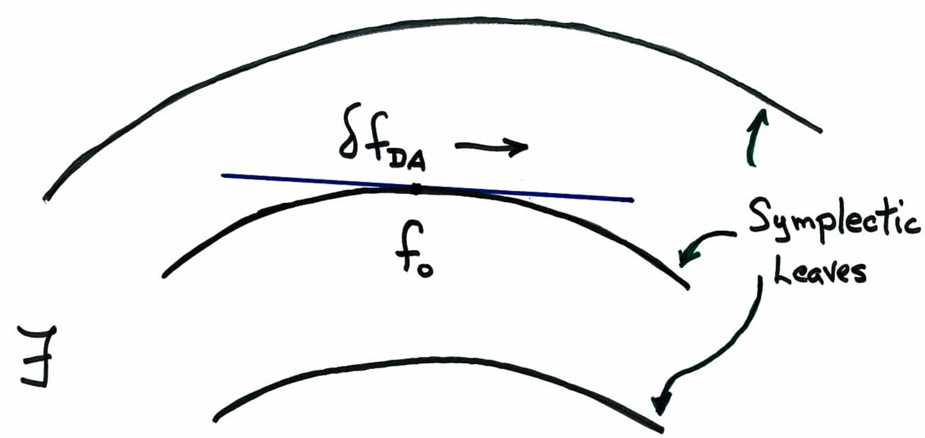
$$\{F,G\}_L = \int f_0 \left[\frac{\delta F}{\delta\delta f}, \frac{\delta G}{\delta\delta f}\right] dx\, dv \ ,$$

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L \,,$$

where quadratic Hamiltonian H_L is the Kruskal-Oberman energy and linear Poisson bracket is $\{\,,\,\}_L = \{\,,\,\}_{f_0}$.

Note:

δf <u>not canonical</u>
H_L <u>not diagonal</u>



Landau's Problem Again

Assume

$$\delta f = \sum_k f_k(v,t)e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t)e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikvf_k + ik\phi_k\frac{e}{m}\frac{\partial f_0}{\partial v} = 0\,, \qquad k^2\phi_k = -4\pi e\int_{\mathbb{R}} f_k(v,t)\,dv$$

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$\begin{aligned} H_L = -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} \, |f_k|^2 \, dv \;\; &+ \;\; \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2 \\ &= \;\; \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) \, \mathcal{O}_{k,k'}(v|v') \, f_{k'}(v') \, dv dv' \end{aligned}$$

Canonization:

$$q_k(v,t) = f_k(v,t) \,, \qquad p_k(v,t) = \frac{m}{ik f_0'} f_{-k}(v,t) \qquad \Longrightarrow$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv$$

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, G[P'_k](v)\, dv$$

Canonical Coordinate Change $(q, p) \longleftrightarrow (Q', P')$:

New Hamiltonian:

$$\begin{aligned} H_L &= \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) \left[Q_k^2(u) + P_k^2(u) \right] \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} d\omega\, \omega\, \frac{|\epsilon(k,\omega)|^2}{\epsilon_I(k,\omega)}\, |E_k(\omega)|^2 = \sum_{k=1}^{\infty} \int_{\mathbb{R}} d\omega\, \omega\, J_k(\omega) \end{aligned}$$

where $\omega_k(u) = |ku|$ and the signature is

$$\boxed{\sigma_k(v) := -\mathrm{sgn}(v f_0'(v))}$$

Note: wave energy (Von Laue 1905) $\sim |E_k(\omega)|^2\, \omega \partial\epsilon/\partial\omega$ has no meaning/use for stable Vlasov continuous spectrum.

- **Continuum Hamiltonian Hopf Bifurcation**

Main CHH Results

- Let f_0 be a stable equilibrium solution of the Vlasov-Poisson equation.

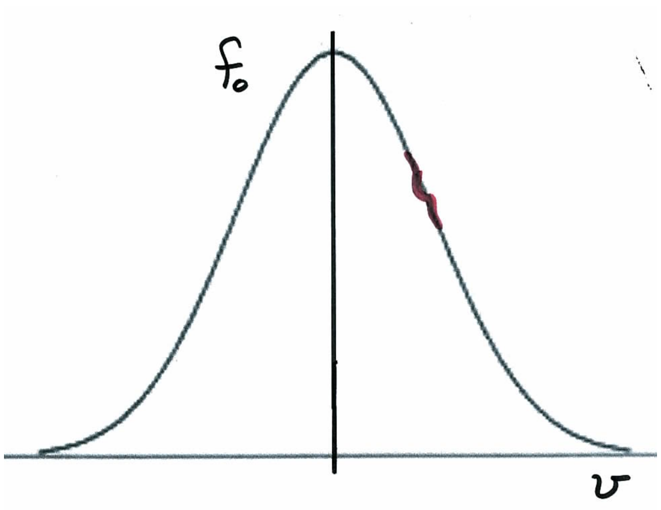
- If $f_0' = 0$ has more than one solution there exist infinitesimal dynamically accessible perturbations that make the system unstable.

- The frequency of the unstable modes is in a neighborhood of the solutions of f_0' that have $f_0'' > 0$.

- If there is only one solution to $f_0' = 0$, then the system is structurally stable.

- If dynamical accessibility is not required then f_0' is always structurally unstable.

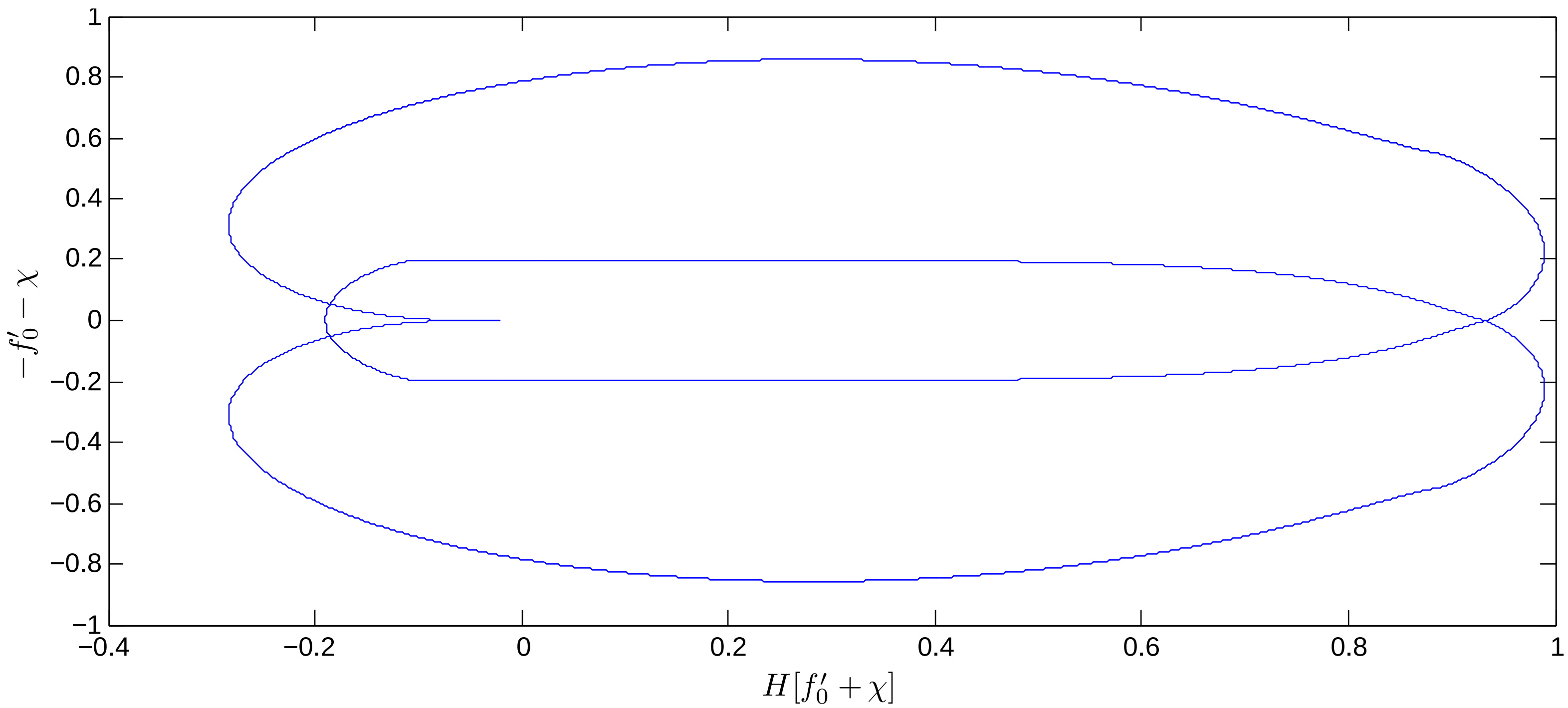
Structurally Unstable Equilibrium



← Pertrubed Maxwellian

Dynamically accessible perturbations are physical perturbations since they result from electric fields.

Destabilization of Maxwellian Distribution



Hamiltonian Spectrum

Hamiltonian Operator:

$$f_{kt} = -ikvf_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v},t) =: T_k f_k\,,$$

Complete System:

$$f_{kt} = T_k f_k \qquad \text{and} \qquad f_{-kt} = T_{-k} f_{-k}\,, \qquad\qquad k \in \mathbb{R}^+$$

Lemma *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and λ^* . Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Spectral Stability

Definition The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is spectrally stable if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Theorem *If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation,*

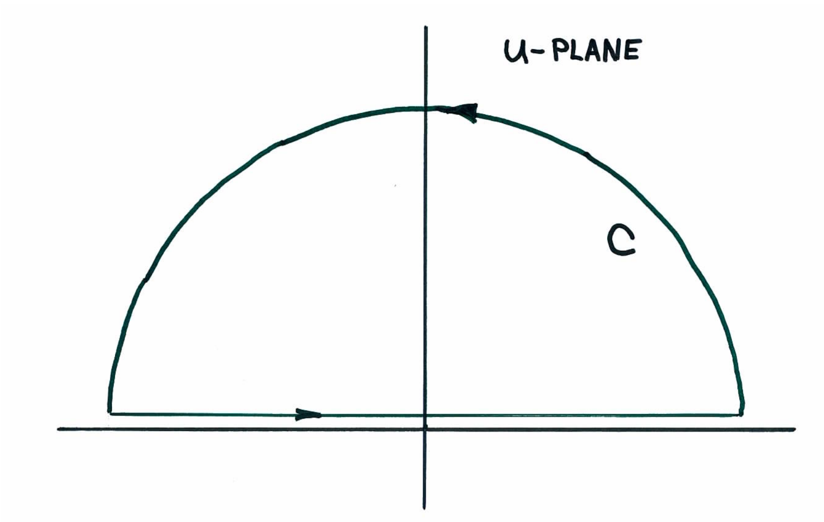
$$\varepsilon(k,u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f_0'}{u - v} = 0 \,,$$

then the system with equilibrium f_0 is spectrally unstable. Otherwise it is spectrally stable.

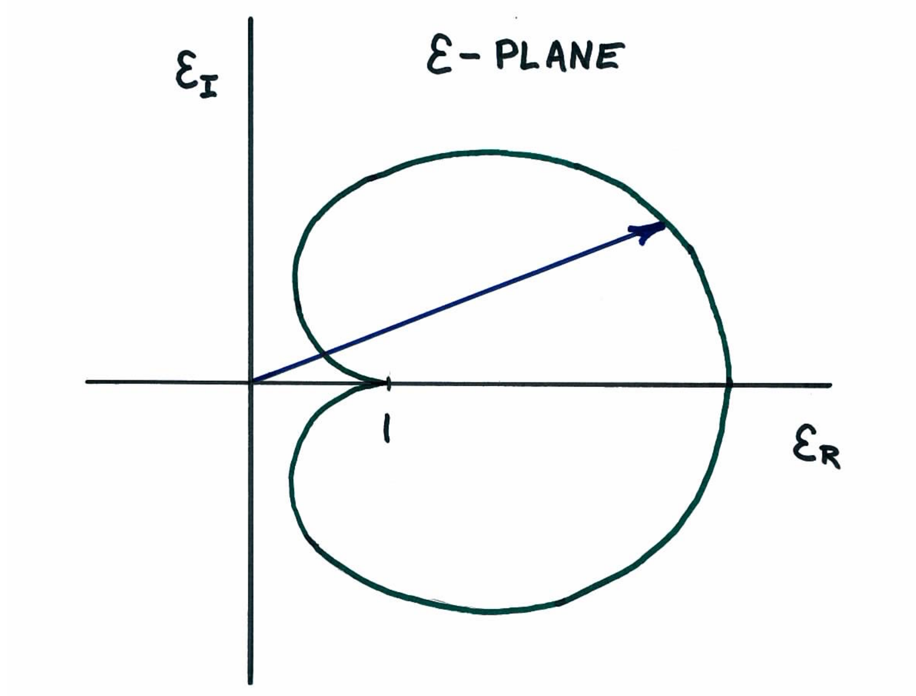
Nyquist Method

$$f_0' \in C^{0,\alpha}(\mathbb{R}) \Rightarrow \varepsilon \in C^{\omega}(uhp).$$

Therefore, Argument Principle $\Rightarrow$ winding # = # zeros of ε



Stable $\rightarrow$



Spectral Theorem

Set $k = 1$ and consider $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions

$$\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$$

Definition Resolvent of T is $R(T,\lambda) = (T - \lambda I)^{-1}$ and $\lambda \in \sigma(T)$. (i) λ in point spectrum, $\sigma_p(T)$, if $R(T,\lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(T)$, if $R(T,\lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(T)$, if $R(T,\lambda)$ exists, densely defined but not bounded.

Theorem *Let* $\lambda = iu$. *(i)* $\sigma_p(T)$ *consists of all points* $iu \in \mathbb{C}$, *where* $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(u - v) = 0$. *(ii)* $\sigma_c(T)$ *consists of all* $\lambda = iu$ *with* $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$. *(iii)* $\sigma_r(T)$ *contains all the points* $\lambda = iu$ *in the complement of* $\sigma_p(T) \cup \sigma_c(T)$ *that satisfy* $f_0'(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.

The CHH Bifurcation

- Usual case: $f_0(v, v_d)$ one-parameter family of equilibria. Vary v_d, embedded mode appears in continuous spectrum, then $\varepsilon(k, \omega)$ has a root in uhp.

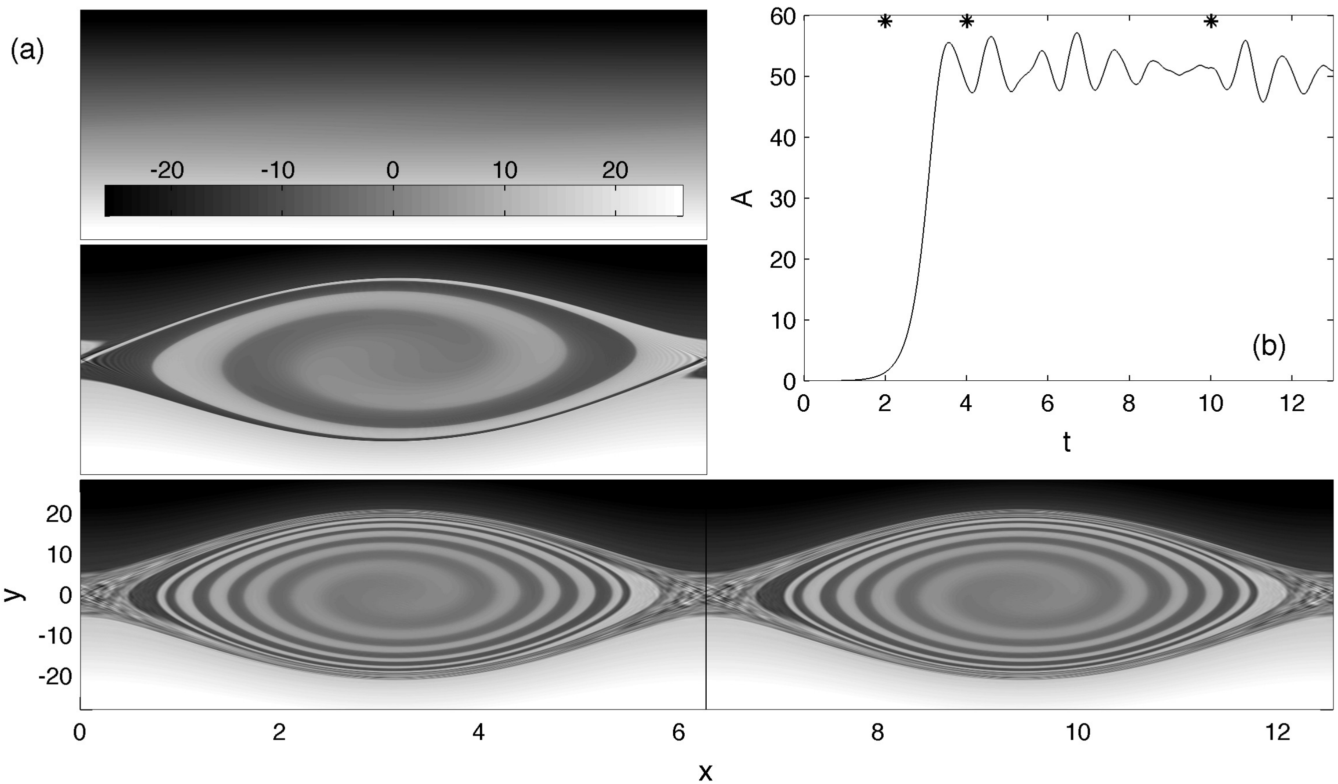
- But all equilibria infinitesmally close to instability in $L^p(\mathbb{R})$. Need measure of distance to bifurcation.

- Waterbag 'onion' replacement for f_0 has ordinary Hamiltonian Hopf bifurcation. Thus, gives a discretization of the continuous spectrum.

- **Nonlinear Normal Forms**

Single-Wave Behavior- Nonlinear

Behavior near marginality in many simulations in various physical contexts



Single-Wave Model

Asymptotics with trapping scaling ... $\Rightarrow$

$$Q_t + [Q, \mathcal{E}] = 0\,, \qquad \mathcal{E} = y^2/2 - \varphi$$

$$\mathrm{i}A_t = \left\langle Q\,\mathrm{e}^{-\mathrm{i}x} \right\rangle\,, \qquad \varphi = A\mathrm{e}^{\mathrm{i}x} + A^*\mathrm{e}^{-\mathrm{i}x}\,,$$

where

$$[f,g] := f_x g_y - f_y g_x\,, \qquad \langle\ \cdot\ \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_0^{2\pi} dx\ \ \cdot \qquad (1)$$

and

$Q(x,y,t)$ = density (vorticity), $\varphi(x,t)$=potential (streamfunction), $A(t)$=single-wave of amplitude, $\mathcal{E}$= particle energy

Model has continuous spectrum with embedded mode that can be pushed into instability and then tracked nonlinearly.

Summary

Underview:

- Integral Transform and Inverse Problem
- Energy and Signature
- Continuum Hamiltonian Hopf Bifurcation
- Nonlinear Normal Forms – Single Wave Model, Hickernell-Berk-Breizman equation, ...

Conclusions

- Useful tool akin to Hilbert or other transforms?
- Applicable to wide class of problems. Tailor to problem.
- Motivates further developments (both physics and math)