Semi-Lagrangian relativistic Vlasov solvers: Key issues and impact on Weibel-type instabilities

Alain Ghizzo[1], M. Sarrat, D. Del Sarto

Institut Jean Lamour- UMR 7198- Université de Lorraine, Nancy, France

alain.ghizzo@univ-lorraine.fr

Topics

1. Vlasov plasmas

2. Reduction techniques in a Hamiltonian framework
   2.1 the multi-stream model
   2.2 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

Introduction (1)

Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m\gamma}.\nabla_x f + \left(\mathbf{E} + \frac{\mathbf{p}}{m\gamma} \times \mathbf{B}\right).\nabla_p f = 0 \qquad \gamma = \sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}}$$

Maxwell equations

$$rot\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad rot\mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{c^2 \partial t}$$

$$div\mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad div\mathbf{B} = 0$$

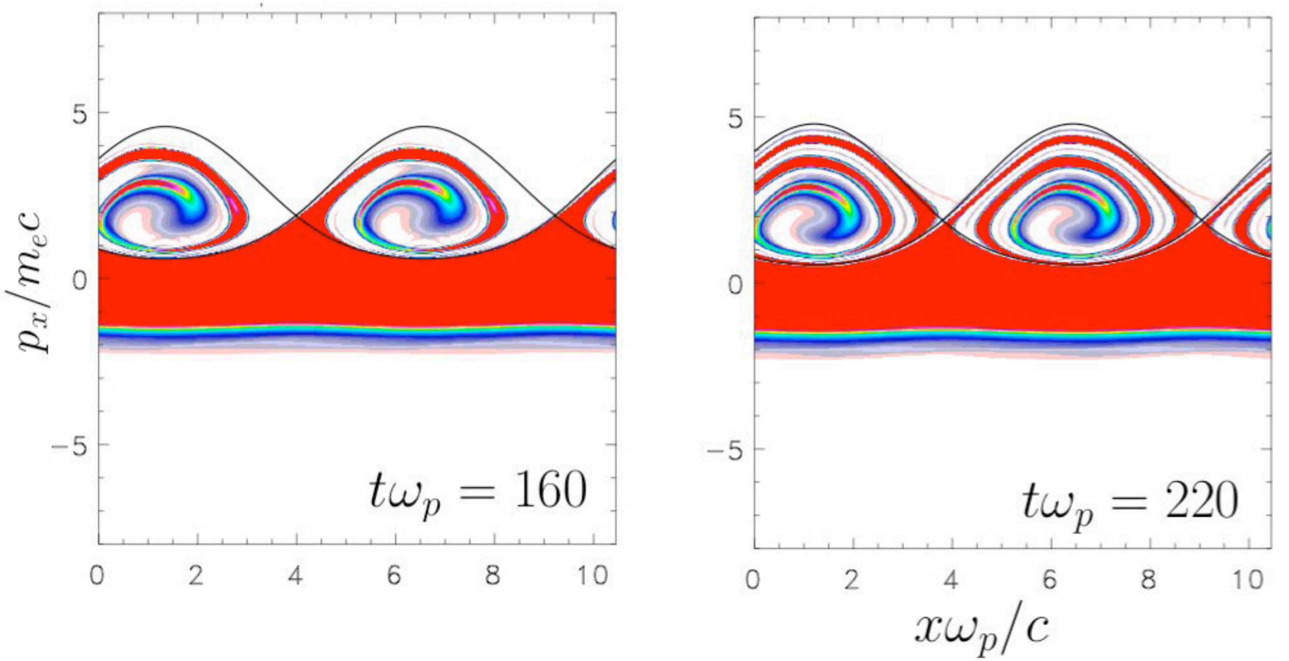
Source terms:

$$\mathbf{J} = \iiint \frac{\mathbf{p}}{m\gamma} f d^3 p \qquad \rho = \iiint f d^3 p$$

Introduction (2)

Vlasov codes: powerful tool for studying in detail the particle dynamics due to

- very fine resolution in phase space
- noiseless character (Raman scattering, BGK,…)



Introduction (3)

. *Questions* for applications:

- Need for a kinetic model? What happens when the phase space dimension is D>5 or even D>6 (general relativity or spin effects)

- Two complementary approaches:
- Need for efficient local advections for parallel algorithms
- Need for Hamiltonian reduction techniques (coupling)

Topics

1. Vlasov plasmas

2. Reduction techniques in a Hamiltonian framework
   2.1 the multi-stream model
   2.2 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

Canonical momemtum invariants (1)

Let us come back to particle motion and consider the Hamiltonian of an electron in an electromagnetic field

$$H = mc^2 \left(1 + \frac{(\mathbf{P}_c - e\,\mathbf{A}_\perp)^2}{m^2 c^2} \right)^{1/2} + e\phi(x,t)$$

Using Coulomb gauge $\nabla \cdot \mathbf{A} = 0 \quad \Rightarrow \quad \mathbf{A} = \mathbf{A}_\perp$

$\mathbf{P}_c = \mathbf{p} + e\mathbf{A}_\perp$ is the canonical momentum

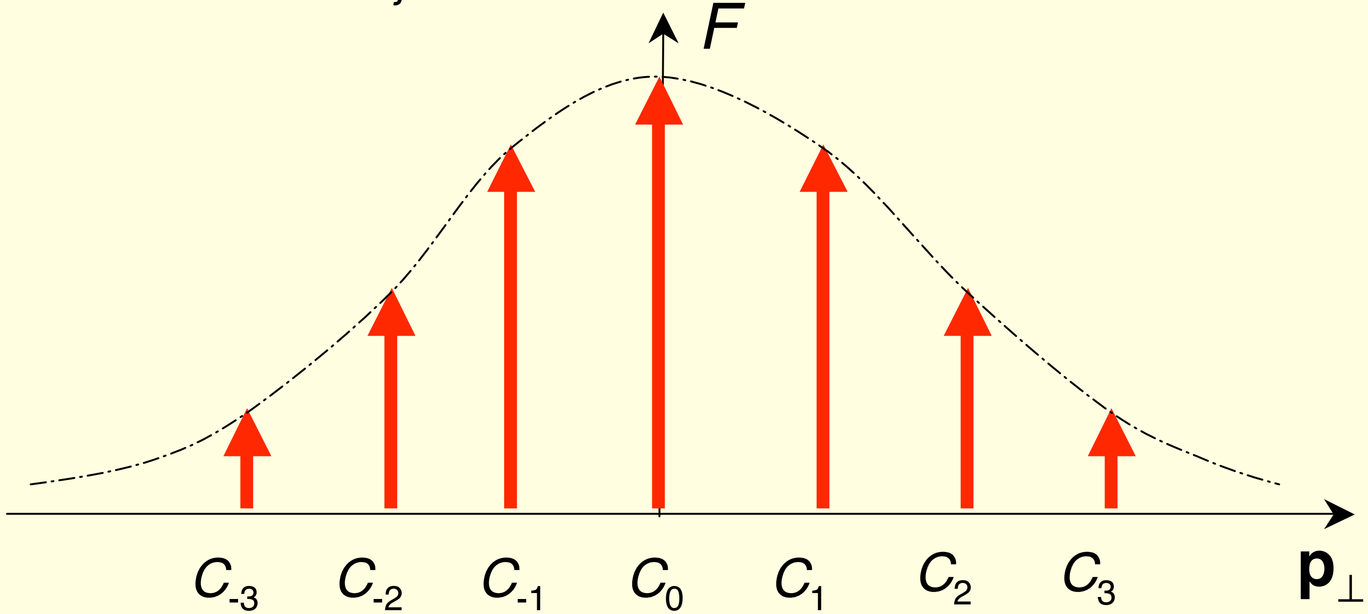
Hamilton equation

$$\begin{cases} \dfrac{dP_{cx}}{dt} = -\dfrac{\partial H}{\partial x} \\ \dfrac{d\mathbf{P}_{c\perp}}{dt} = -\nabla_\perp H = 0 \quad \Rightarrow \quad \mathbf{P}_{c\perp} = const \end{cases}$$

Caconical momentum invariants (2)

The full distribution function can be written as a sum of Dirac delta masses

$$F(x,p_x,\mathbf{p}_\perp,t) = \sum_j f_j(x,p_x,t)\,\delta\left[\mathbf{p}_\perp - \left(\mathbf{C}_j - e\,\mathbf{A}_\perp(x,t)\right)\right]$$



Caconical momentum invariants (3)

Now let us divide the electron population into a finite number j_{max} of groups j

Each group has the same perpendicular constant canonical momentum $\mathbf{C}_j$

The hamiltonian of the j-particles is

$$H_j = mc^2(\gamma_j - 1) + e\phi(x,t)$$

With the Lorentz factor

$$\gamma_j(x,p_x,t) = \left[1 + \frac{p_x^2 + \left(\mathbf{C}_j - e\mathbf{A}_\perp(x,t)\right)^2}{m^2c^2}\right]^{1/2}$$

The multi-stream (MS) model (1)

Several Vlasov equations $\mathbf{P}_{C\perp} = \mathbf{p} + e\mathbf{A}_\perp = const = \mathbf{C}_j$

$$\frac{\partial f_j}{\partial t} + \frac{p_x}{m\gamma_j}\frac{\partial f_j}{\partial x} + \left(eE_x - \frac{1}{2m\gamma_j}\frac{\partial\left(C_j - e\mathbf{A}_\perp\right)^2}{\partial x}\right)\frac{\partial f_j}{\partial p_x} = 0$$

Lorentz factor

$$\gamma_j = \sqrt{1 + \frac{p_x^2}{m^2c^2} + \left(\frac{C_j - eA_\perp(x,t)}{mc}\right)^2}$$

Coupled to:

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2\frac{\partial^2 \mathbf{A}_\perp}{\partial x^2} = \frac{1}{\varepsilon_0}\sum_{j=-N}^{+N}\mathbf{J}_{\perp j}(x,t) \quad \text{and} \quad \frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}\left(\sum_{j=-N}^{+N} n_j(x,t) - n_0\right)$$

MS model (2)

Several Vlasov equations $\mathbf{P}_{C\perp} = \mathbf{p} + e\mathbf{A}_\perp = const = \mathbf{C}_j$

$$\frac{\partial f_j}{\partial t} + \frac{p_x}{m\gamma_j}\frac{\partial f_j}{\partial x} + \left(eE_x - \frac{1}{2m\gamma_j}\frac{\partial \left(C_j - e\mathbf{A}_\perp\right)^2}{\partial x} \right)\frac{\partial f_j}{\partial p_x} = 0$$

Source terms

$$n_j(x,t) = \int_{-\infty}^{+\infty} f_j(x,p_x,t)\,dp_x$$

$$J_{\perp j} = \frac{e}{m}\left(C_j - eA_\perp\right)\int_{-\infty}^{+\infty}\frac{f dp_x}{\gamma_j}$$

Coupled to:

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2\frac{\partial^2 \mathbf{A}_\perp}{\partial x^2} = \frac{1}{\varepsilon_0}\sum_{j=-N}^{+N}\mathbf{J}_{\perp j}(x,t) \quad \text{and} \quad \frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}\left(\sum_{j=-N}^{+N} n_j(x,t) - n_0\right)$$

MS model: linear analysis of WI or CFI (1)

. *Questions* for applications to Weibel Instability (WI) and Current Filamentation Instability (CFI)

- Both instabilities are considered as basic processes in plasmas associated with the generation of magnetic fields
- The multi-stream model allows to « unify » CFI and WI in a global approch

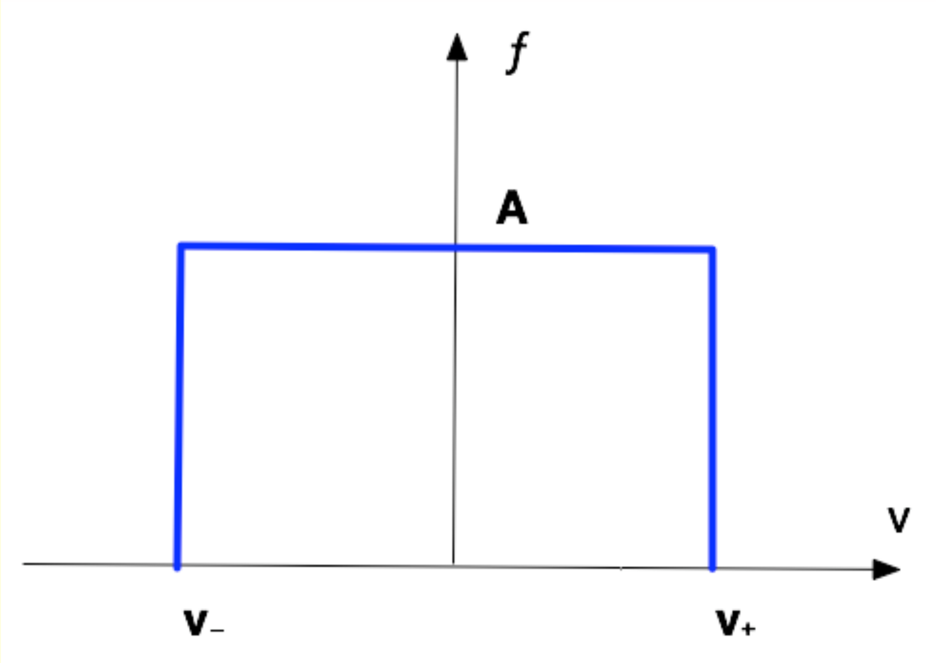
Water-Bag approach in p_x and multi-stream approach in $\mathbf{p}_\perp$:

$$f(x,\mathbf{p},t) = \sum_{j=-N}^{+N} F_j \left[\mathrm{H}\left(p_x + p_j^+\right) - \mathrm{H}\left(p_x + p_j^+\right) \right] \delta\left(\mathbf{p}_\perp - \left(\mathbf{C}_j - e\mathbf{A}_\perp\right)\right)$$

MS model: linear analysis of WI or CFI (2)

The distribution function may be described as:

H is the Heaviside fstep function and $+p_m$ and $-p_m$ are the limit values in momentum



modified Lorentz factor $$\gamma_j^{\pm} = \sqrt{1 + \frac{p_x^{\pm 2}}{m^2 c^2} + \left(\frac{C_j - eA_\perp(x,t)}{mc} \right)^2}$$

MS model: linear analysis of WI or CFI (3)

$$\frac{dp_j^\pm}{dt}=\frac{\partial p_j^\pm}{\partial t}+\frac{p_j^\pm}{m\gamma_j^\pm}\frac{\partial p_j^\pm}{\partial x}=eE_x-\frac{1}{2m\gamma_j^\pm}\frac{\partial}{\partial x}\left(\mathbf{C}_j-e\mathbf{A}_\perp\right)^2$$

Longitudinal field
$$\frac{\partial E_x}{\partial x}=\frac{e}{\varepsilon_0}\left(\sum_{j=-N}^{+N}n_j(x,t)-n_0\right)$$

Transverse potential vector
$$\frac{\partial^2\mathbf{A}_\perp}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}_\perp}{\partial x^2}=\frac{1}{\varepsilon_0}\sum_{j=-N}^{+N}\mathbf{J}_{\perp j}(x,t)$$

Source terms

$$J_{\perp j}=\frac{e}{m}\left(C_j-eA_\perp\right)\int_{-\infty}^{+\infty}\frac{f dp_x}{\gamma_j}$$

and

$$n_j(x,t)=\int_{-\infty}^{+\infty}f_j(x,p_x,t)dp_x$$

MS model: linear analysis of WI or CFI (4)

Normal modes analysis around an equilibrium:

$$p_j^{\pm} = \pm p_{aj} + \delta p_j^{\pm}$$

$$n_j = n_{0j} + \delta n_j$$

and

$$E_x = \delta E_x \qquad \text{and} \qquad \mathbf{A}_\perp = \delta \mathbf{A}_\perp$$

MS model: linear analysis of WI or CFI (5)

$$\frac{\partial \delta p_j^{\pm}}{\partial t} \pm \frac{p_{aj}}{m\Gamma_{aj}} \frac{\partial \delta p_j^{\pm}}{\partial x} = e\delta E_x + \frac{e}{m\Gamma_{aj}} \mathbf{C}_j . \partial_x \delta \mathbf{A}_{\perp}$$

Longitudinal field

$$\frac{\partial \delta E_x}{\partial t} = \frac{e}{\varepsilon_0} \sum_{j=-N}^{+N} F_j \left(\delta p_j^+ - \delta p_j^- \right)$$

potential vector

$$\frac{\partial^2 \mathbf{A}_{\perp}}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{A}_{\perp}}{\partial x^2} = \frac{e}{m\varepsilon_0} \sum_{j=-N}^{+N} \left(\mathbf{C}_j - e\delta \mathbf{A}_{\perp} \right) \rho_j$$

with

$$\rho_j = \int_{-\infty}^{+\infty} \frac{f_j}{\gamma_j} dp_x = mcF_j \ln \left(\frac{u_j^+ + \gamma_j^+}{u_j^- + \gamma_j^-} \right)$$ and $$\Gamma_{aj} = \sqrt{1 + \frac{p_{aj}^2}{m^2 c^2} + \left(\frac{\mathbf{C}_j}{mc} \right)^2}$$

MS model: linear analysis of WI or CFI (6)

General dispersion relation coonected to a matrix of type:

$$[D] = \begin{pmatrix} D_{xx} & D_{xy} & 0 \\ D_{xy} & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{pmatrix}$$

Coupling introduced by a disymmetry due to relativity and/or WB

MS model: linear analysis of WI or CFI (7)

We introduce a normalized density of « beam » j :

$$\alpha_j = \frac{2 p_{aj} F_j}{n_0}$$

We impose the following conditions

$$\sum_{j=-N}^{+N} \alpha_j = 1 \quad \text{and} \quad \sum_{j=-N}^{+N} \alpha_j C_j \left\langle \gamma_{aj}^{-1} \right\rangle = 0$$

MS model: linear analysis of WI or CFI (8)

Dispersion relation of type:

$$\left[\omega^2-\omega_k^2+\omega_p^2\left(\omega^2-k^2c^2\right)\sum_{j=-N}^{+N}\frac{\alpha_j C_j^2}{m^2c^2\Gamma_{aj}\left(1+\frac{C_j^2}{m^2c^2}\right)\left(\omega^2-k^2v_{aj}^2\right)}\right]$$

$$\left[1-\omega_p^2\sum_{j=-N}^{+N}\frac{\alpha_j}{\Gamma_{aj}\left(\omega^2-k^2v_{aj}^2\right)}\right]=k^2\omega_p^4\left[\sum_{j=-N}^{+N}\frac{\alpha_j C_j}{m\Gamma_{aj}^2\left(\omega^2-k^2v_{aj}^2\right)}\right]^2$$

Coupling by Assymmetry

With $$\omega_k^2=\omega_p^2\sum_{j=-N}^{+N}\alpha_j\left\langle\gamma_{aj}^{-1}\right\rangle+k^2c^2$$

Topics

1. Vlasov plasmas

2. Reduction techniques in a Hamiltonian framework
   2.1 the multi-stream model
   2.2 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

Description of CFI using full Pressure tensor dynamics: j=2

Conservation density for beam + plasma
$$\frac{\partial n_j}{\partial t} + \nabla_x.(\mathbf{u}_j n_j) = 0$$

Others moments of f
$$\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{u}_j.\nabla \mathbf{u}_j = \frac{e}{m}(\mathbf{E} + \mathbf{u}_j \times \mathbf{B}) - \frac{\nabla.\Pi_j}{mn_j}$$

Pressure tensor
$$\frac{\partial \Pi_j}{\partial t} + \nabla.(\mathbf{u}_j \Pi_j) + \nabla \mathbf{u}_j.\Pi_j + (\nabla \mathbf{u}_j.\Pi_j)^T$$
$$= \frac{e}{m}\left(\Pi_j \times \mathbf{B} + (\Pi_j \times \mathbf{B})^T\right) - \nabla.\mathbf{Q}$$

and
$$Q = mn_j \langle (\mathbf{v} - \mathbf{u}_j)(\mathbf{v} - \mathbf{u}_j)(\mathbf{v} - \mathbf{u}_j) \rangle_j$$

zero

Tensor Pressure model: linear analysis of CFI (1)

Equilibrium of type (anisotropic pressure tensor):

$$\Pi^{(0)} = \begin{pmatrix} \Pi_{xx}^{(0)} & 0 & 0 \\ 0 & \Pi_{yy}^{(0)} & 0 \\ 0 & 0 & \Pi_{zz}^{(0)} \end{pmatrix}$$

« thermal » velocities:

$$c_{x,j} = \sqrt{\frac{\Pi_{xx,j}^{(0)}}{m n_j^{(0)}}} \qquad c_{y,j} = \sqrt{\frac{\Pi_{yy,j}^{(0)}}{m n_j^{(0)}}}$$

Imposed conditions:

$$\mathbf{J}^{(0)} = e\sum_{j=1}^{2} n_j^{(0)} \mathbf{u}_j^{(0)} = 0 \qquad \sum_{j=1}^{2} n_j^{(0)} = n_0$$

Tensor Pressure model: linear analysis of CFI (2)

General dispersion relation coonected to a matrix of type:

$$[D].\mathbf{E} = 0 \qquad [D] = \begin{pmatrix} D_{xx} & D_{xy} & 0 \\ D_{xy} & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{pmatrix}$$

WI- CFI coupling by beam assymetry

Tensor Pressure model: linear analysis of CFI (3)

dispersion relation

$$\left[1-\frac{k^2c^2}{\omega^2}-\sum_{j=-N}^{+N}\omega_{p,j}^2\left(\frac{\omega^2+k^2\left(c_{y,j}^2-c_{x,j}^2\right)}{\omega^2-k^2c_{x,j}^2}-\frac{\left(ku_{y,j}^{(0)}\right)^2}{\omega^2-3k^2c_{x,j}^2}\right)\right]$$

$$\left[1-\sum_{j=1}^{2}\frac{\omega_{p,j}^2}{\left(\omega^2-3k^2c_{x,j}^2\right)}\right]=\left[\sum_{j=-N}^{+N}\frac{\omega_{p,j}^2ku_{y,j}^{(0)}}{\omega\left(\omega^2-k^2v_{aj}^2\right)}\right]^2$$

WI- CFI coupling by beam assymetry

$$D_{xy}=-\frac{\omega^2}{c^2}\sum_{j=1}^{2}\frac{ku_{y,j}^{(0)}}{\omega}\frac{\omega_{p,j}^2}{\left(\omega^2-k^2v_{aj}^2\right)}$$

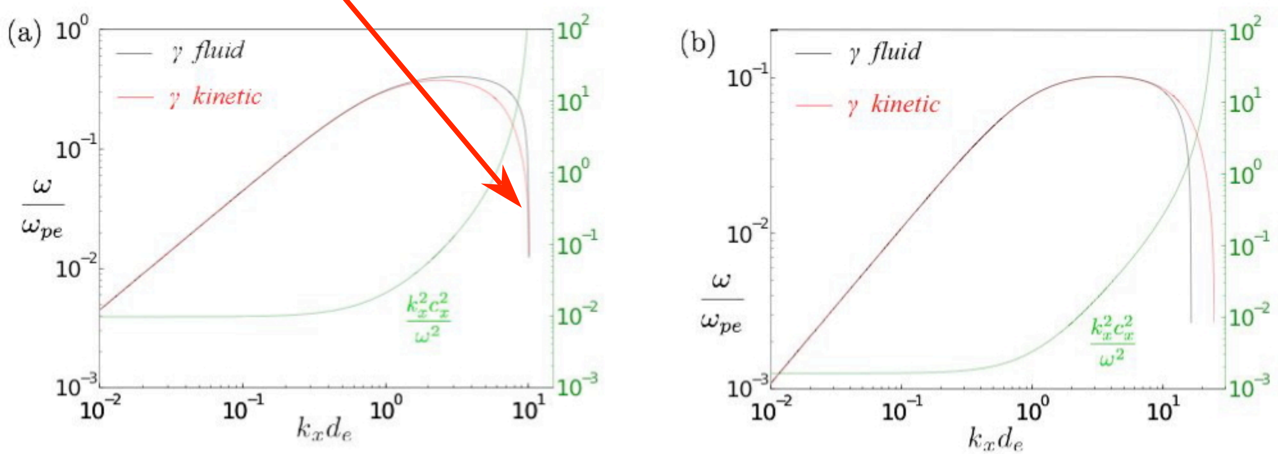
The coupling disppears when: $\Pi_{xx,1}^{(0)}u_{y,1}^{(0)}+\Pi_{xx,2}^{(0)}u_{y,2}^{(0)}=0$

Tensor Pressure model: linear analysis of CFI (4)

$$\left[1-\frac{k^2c^2}{\omega^2}-\sum_{j=-N}^{+N}\omega_{p,j}^2\left(\frac{\omega^2+k^2\left(c_{y,j}^2-c_{x,j}^2\right)}{\omega^2-k^2c_{x,j}^2}-\frac{\left(ku_{y,j}^{(0)}\right)^2}{\omega^2-3k^2c_{x,j}^2}\right)\right]$$

$$\left[1-\sum_{j=1}^{2}\frac{\omega_{p,j}^2}{\left(\omega^2-3k^2c_{x,j}^2\right)}\right]=\left[\sum_{j=-N}^{+N}\frac{\omega_{p,j}^2 ku_{y,j}^{(0)}}{\omega\left(\omega^2-k^2v_{aj}^2\right)}\right]^2$$

Cut-off wave-number recovered



Work in progress: Poster of M. Sarrat, Oral Comm: D. Del Sarto In the Vlasovia Conference

Topics

1. Vlasov plasmas

2. Reduction techniques in a Hamiltonian framework
   2.1 the multi-stream model
   2.2 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

An Example: 1D2V SL Vlasov-Maxwell Simulation for WI

We consider a 1D2V phase space using an electron distribution function of type:

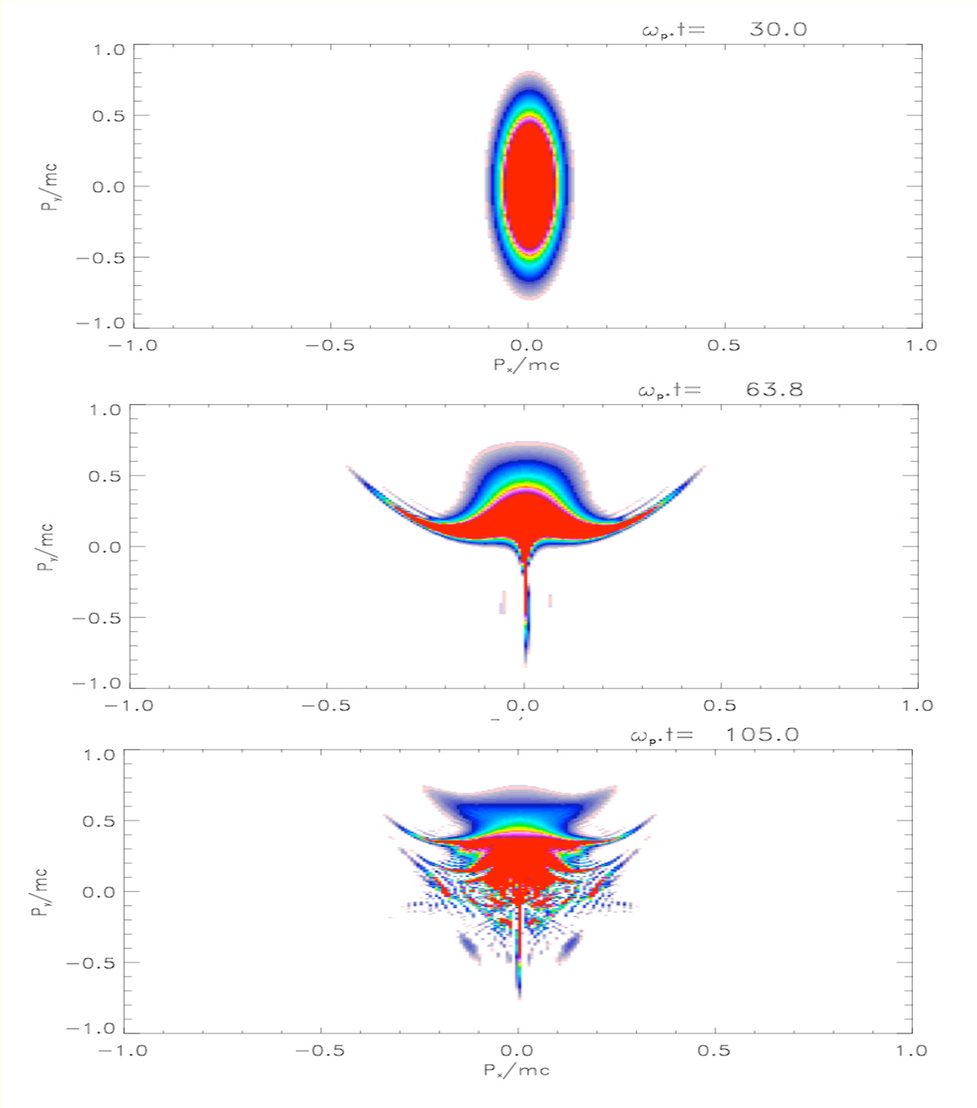
$$f = f(x, p_x, p_y, t)$$

We simulate the Weibel instability using a Backward Semi-Lagrangian scheme for the resolution of the Vlasov equation in 1D2V phase space

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m\gamma}.\nabla_x f + \left(\mathbf{E} + \frac{\mathbf{p}}{m\gamma} \times \mathbf{B}\right).\nabla_p f = 0$$

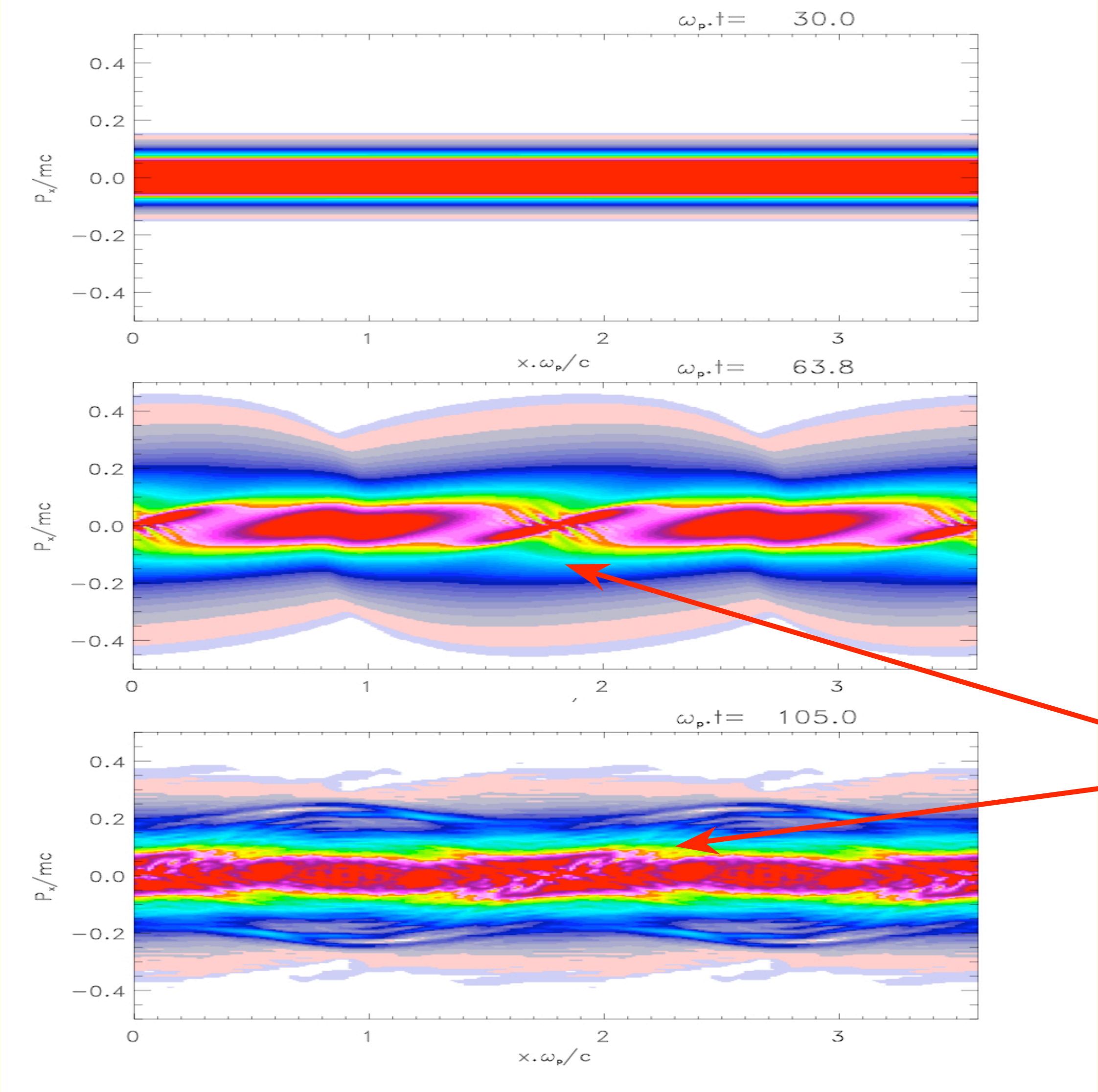
For an anisotropy in temperature 1kev in Px and 50keV in Py

1D2V Vlasov-Maxwell simulation (1)



Phase space Px-Py overview

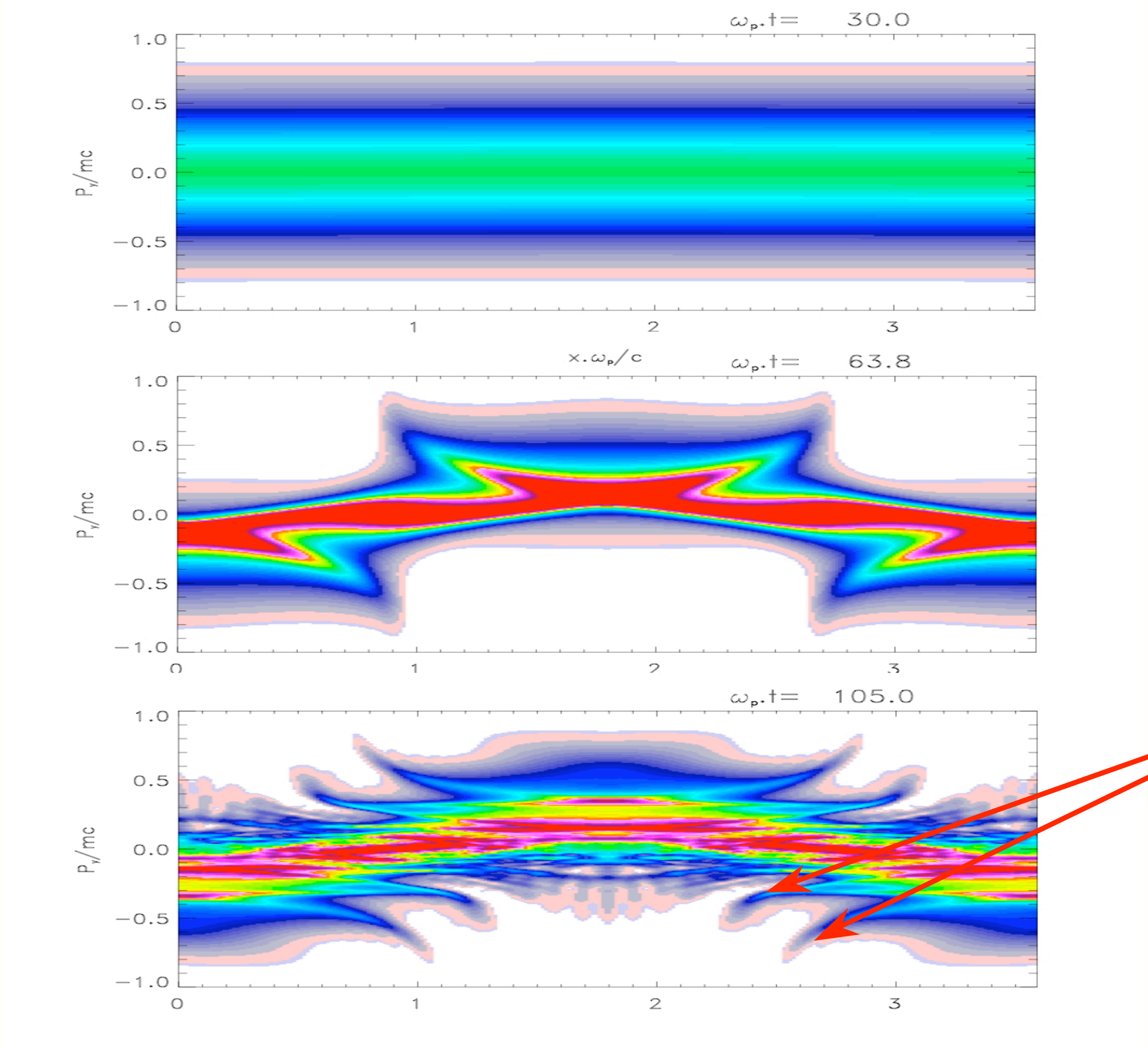
1D2V Vlasov-Maxwell simulation (2)



Phase space x-Px overview

Complex behavior even in a simple case with 3D phase space

1D2V Vlasov-Maxwell simulation (3)



Phase space x-Py overview

Complex behavior even in a simple case with 3D phase space:

Suggests that several « beams » have been selected by the plasma of different magnetic bounce frequencies

Using canonical invariants as diagnostics (1)

We consider a 1D2V phase space using an electron distribution function of type:

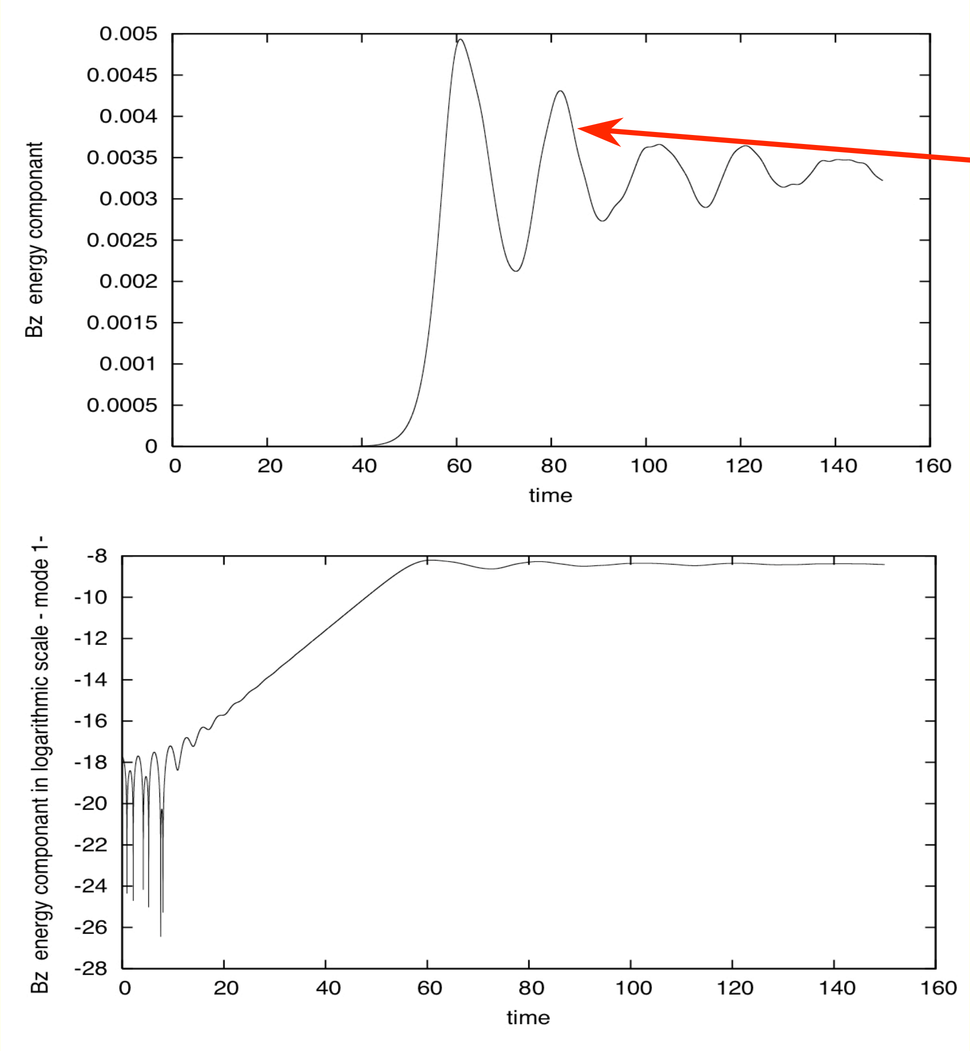
$$f_j(x, p_x, t)$$

We simulate the Weibel instability using a multi-stream model with 7 streams

$$\frac{\partial f_j}{\partial t} + \frac{p_x}{m\gamma_j}\frac{\partial f_j}{\partial x} + \left(eE_x - \frac{1}{2m\gamma_j}\frac{\partial \left(C_j - e\mathbf{A}_\perp \right)^2}{\partial x} \right)\frac{\partial f_j}{\partial p_x} = 0$$

Coupled, in a self-consistent way, with Maxwell equations

Using canonical invariants as diagnostics (2)



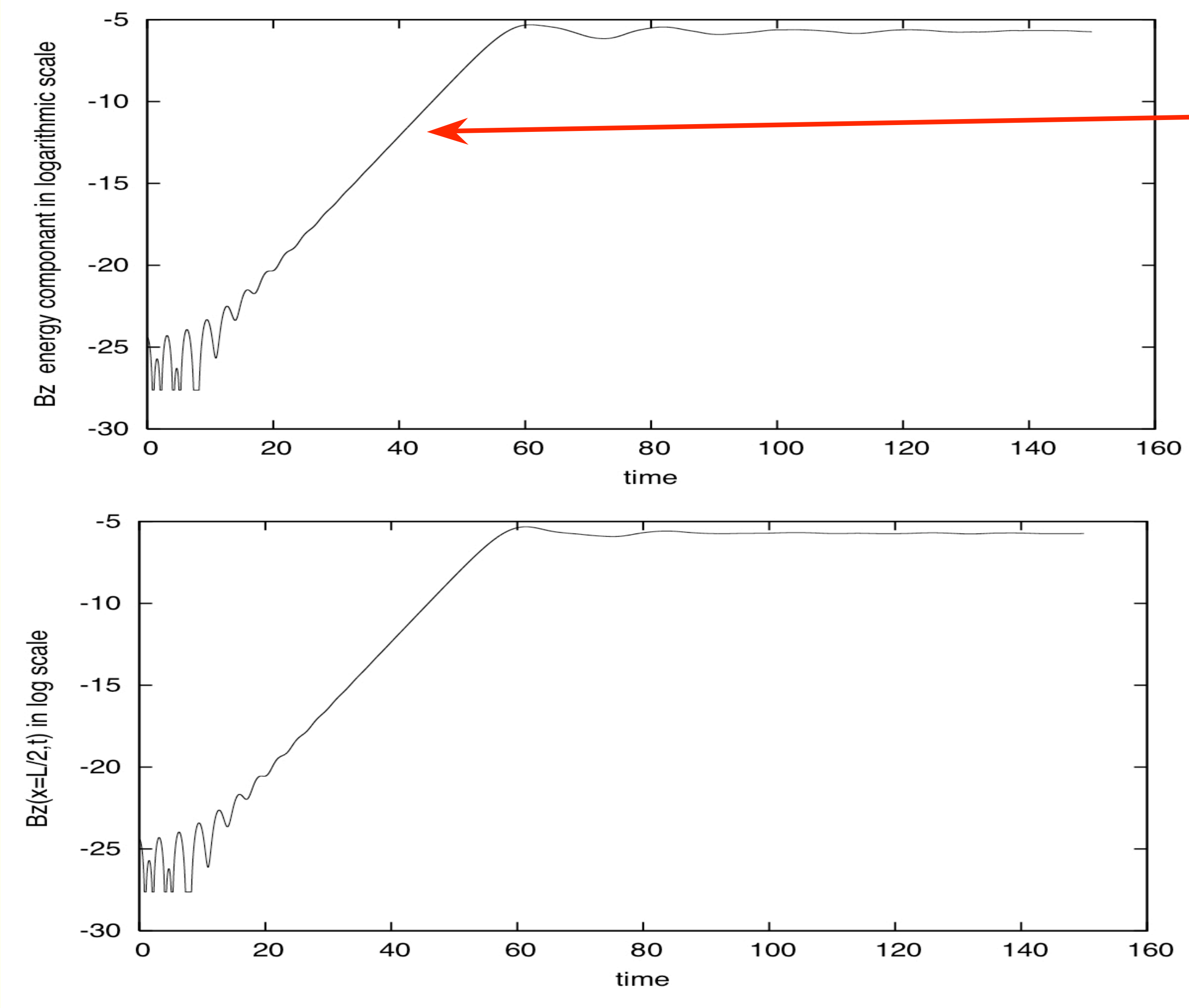
Multi-stream: magnetic bounce frequency in good agreement

$$\frac{\omega_b}{\omega_p} = \sqrt{\frac{kc}{\omega_p}\frac{p_\perp}{mc}\frac{eB_{\max}}{m\omega_p}} \approx 0.31$$

Multi-stream: growthrate is found in good agreement

$$\frac{\gamma}{\omega_p} \approx 0.40$$

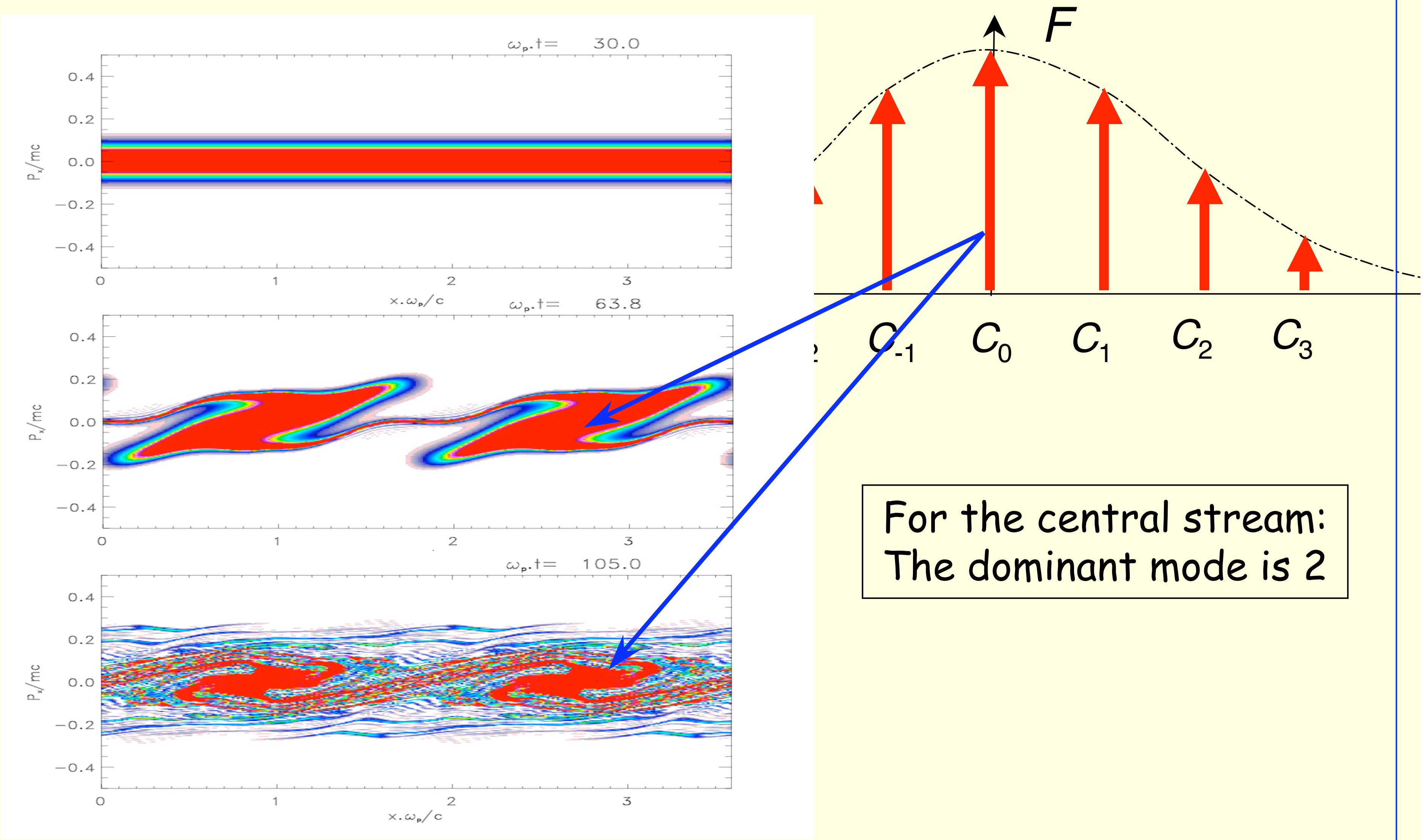
Using canonical invariants as diagnostics (3)



SL-VM solver:

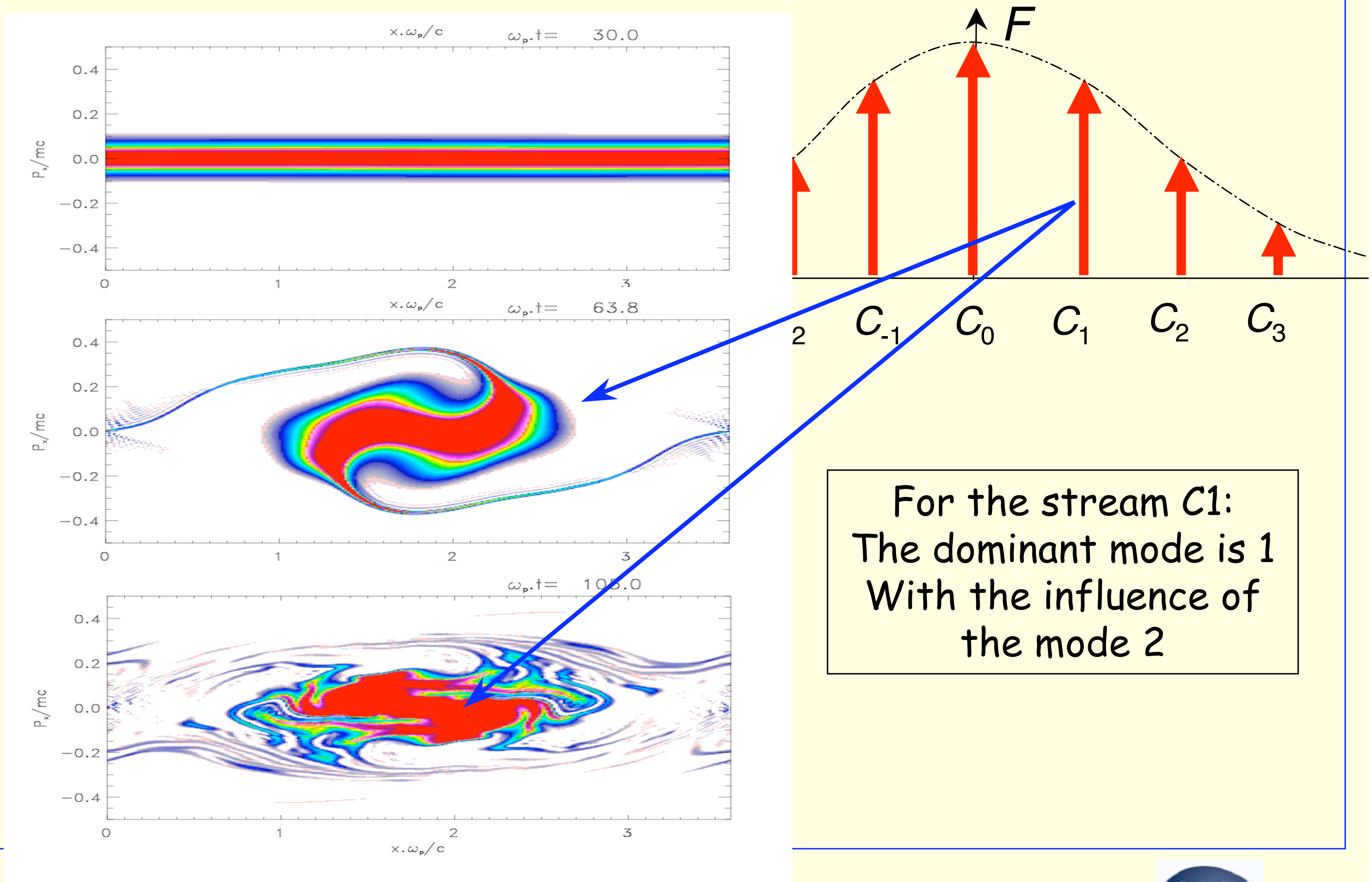
Multi-stream solver

Using canonical invariants as diagnostics (4)



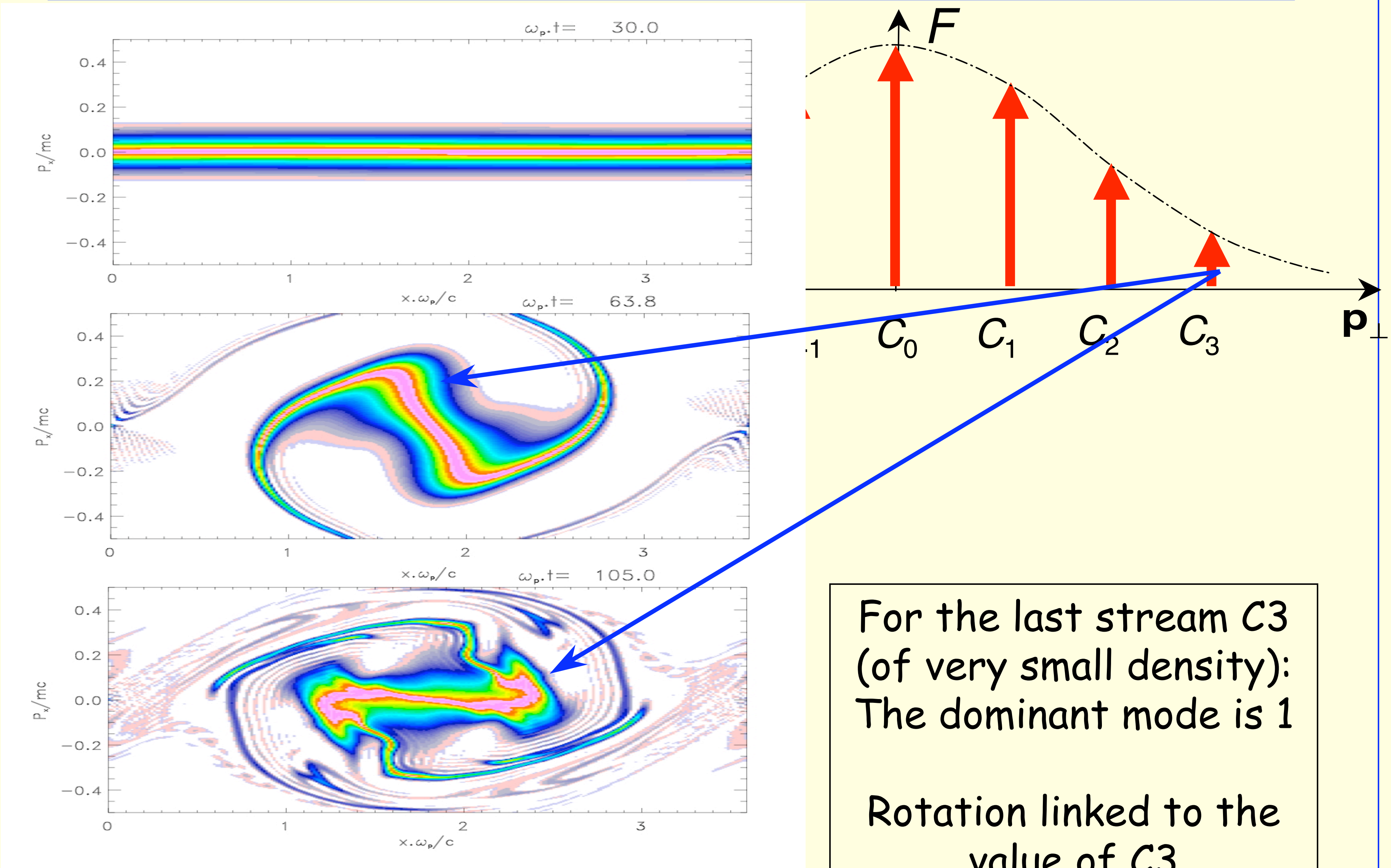
For the central stream:
The dominant mode is 2

Using canonical invariants as diagnostics (5)



For the stream C1:
The dominant mode is 1
With the influence of
the mode 2

Using canonical invariants as diagnostics (6)



For the last stream C3 (of very small density): The dominant mode is 1

Rotation linked to the value of C3

Topics

1. Vlasov plasmas

2. Reduction techniques in a Hamiltonian framework
   2.1 the multi-stream model
   2.2 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

Conclusions

Semi-Lagrangian Vlasov codes applied for the study of CFI and WI in the relativistic regime

- lack of numerical noise
- good resolution in phase space

The multi-stream model is a set of kinetic Vlasov-type equations obtained in a Hamiltonian framework allowing to reduce the dimension of phase space

Two complementary approaches: SL VM solver and multi-stream model

Comparison PIC-Vlasov (1)

Sampling the x-space needs $\sim (L/\lambda_D)^{d_x}$ d_x : real space dimension

PIC Codes

$$N_{part} = n_0 (L)^{d_x}$$

$$\sim n_0 (\lambda_D)^{d_x} \left(\frac{L}{\lambda_D}\right)^{d_x}$$

$$\sim \frac{1}{g_{pic}} (L/\lambda_D)^{d_x}$$

$$g_{pic} = \left(n_0 \lambda_D^{d_x}\right)^{-1}$$

is the graininess due to particules

Vlasov Codes

Real space X momentum space

$$N_{vlas} \sim (L/\lambda_D)^{d_x} N_v^{d_v}$$

$$\sim g_{pic} N_{part} N_v^{d_v}$$

d_v : momentum space dimension

N_v : sampling of momentum space in each direction

Comparison PIC-Vlasov (2)

Assume the same CPU time
•to push a particle (PIC)
•to move a phase space mesh point (Vlasov)

$$\frac{CPU_{vlas}}{CPU_{pic}} = g_{pic} N_v^{d_v}$$

The ratio of the computationnal effort between Vlasov and PIC depends on
•PIC graininess (must be as low as possible)
•Sampling of momentum space (must be as high as possible)

Comparison PIC-Vlasov (3)

$\frac{N_{vlas}}{N_{part}}$	$D_v=1$	$D_v=2$	$D_v=3$
$g_{PIC}=10^{-2}$	1	100	10 000
$g_{PIC}=10^{-4}$	0.01	1	100
$g_{PIC}=10^{-6}$	0.0001	0.01	1

Prefer PIC

Prefer Vlasov

What happens when g_{PIC} smaller
Availaible today?

Complementary Approach

1. Vlasov plasmas

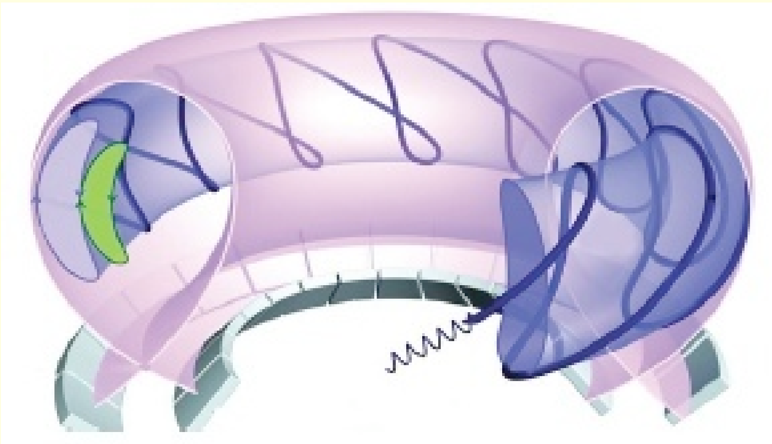
2. Reduction techniques in a Hamiltonian framework
   2.1 adiabatic invariants
   2.2 the multi-stream model
   2.3 Pressure tensor dynamics

3. Application to the Weibel-type instabilities

4. Conclusions

Adiabatic invariants (1)

• Each adiabatic invariant reduces the dimentionality by a factor two and a trapped particle is fully described by the position of the « banana center » (action-angle variables)



$$\omega_k = \frac{d\phi_k}{dt} = \frac{\partial H}{\partial J_k}$$

$$\frac{dJ_k}{dt} = 0 \quad for \quad k = 1,2$$

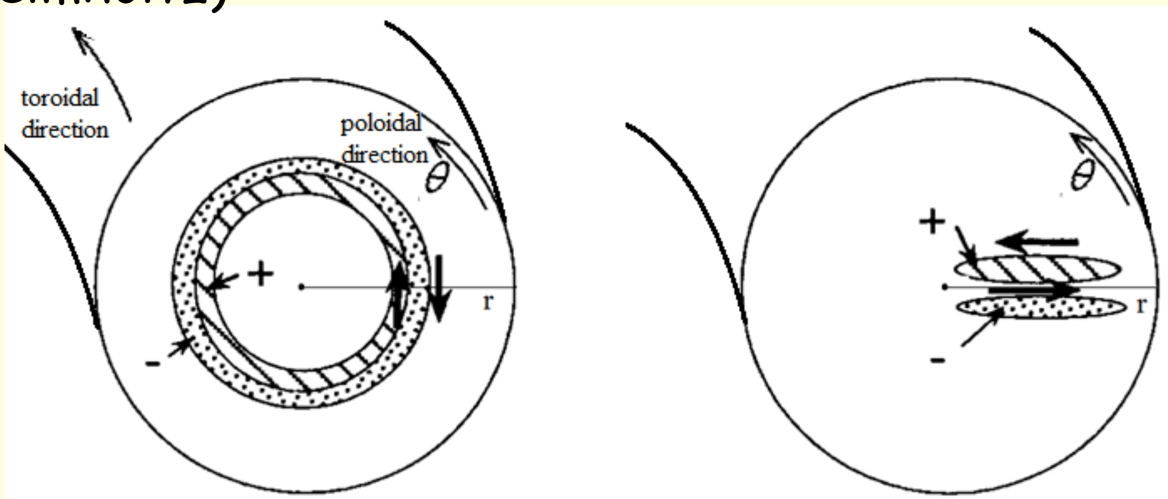
In laser-plasma interaction: No really equivalent approach although BGK waves (EAWs, KEEN waves) seem to play a major role

Adiabatic invariants (2)

$$\frac{\partial f_{\kappa,E}}{\partial t}+\omega_d(\kappa;s)E\frac{\partial f_{\kappa,E}}{\partial \phi_3}+\frac{\partial J_0 U}{\partial \psi}\frac{\partial f_{\kappa,E}}{\partial \phi_3}-\frac{\partial J_0 U}{\partial \phi_3}\frac{\partial f_{\kappa,E}}{\partial \psi}=0$$

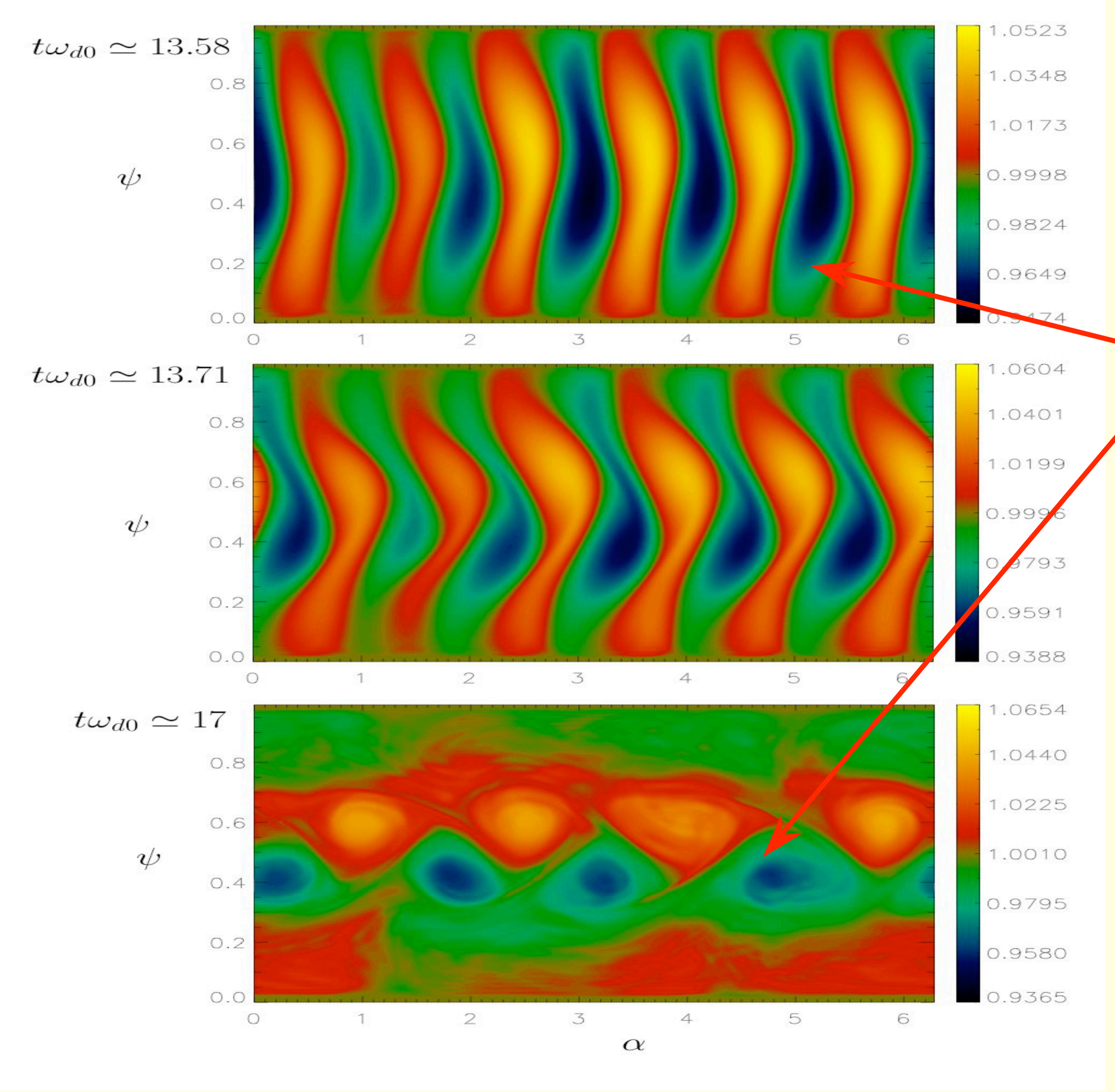
Which is coupled with $C\left\{U-\langle U\rangle_{\psi}\right\}-C\alpha\nabla^2 U=\bar{n}_i(\phi_3,\psi,t)-1$

The model allows us to study the coupling between streamers, zonal flows et other types of structures acting in turbulence (Kelvin-Helmholtz)



Zonal flow streamer

Adiabatic invariants (3)



« transition » observed due to the presence of KH

Turbulence induced by trapped ion modes (TIM)