Nonlinear hydrid simulations of precessional Fishbone instability

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Outline

Context: the Fishbone instability

- First observation on the PDX tokamak, with near perpendicular neutral beam injection active
- Bursts of electromagnetic instability , associated with losses of fast particles
- Mode frequency consistent with trapped fast ion precession frequency
- Dominant $m = n = 1$, kink-shaped, mode structure
- Observation of frequency chirping

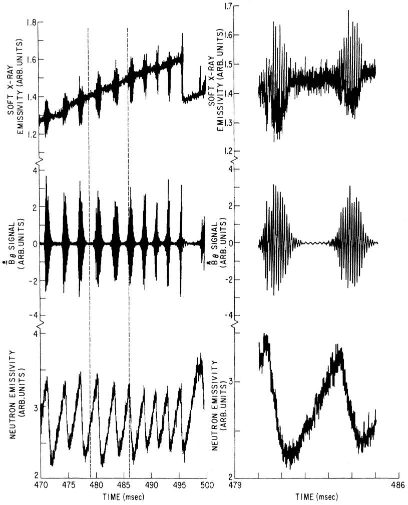
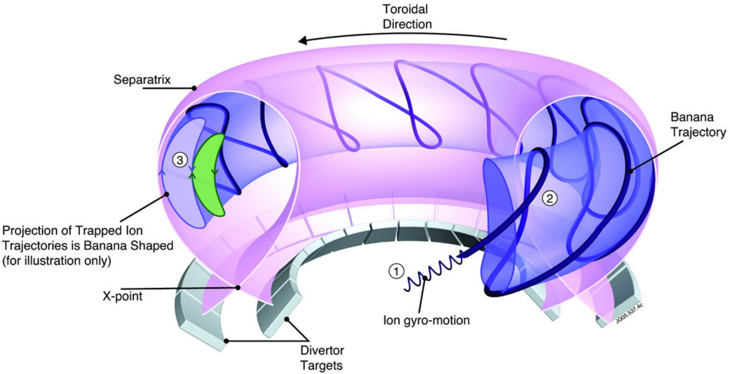


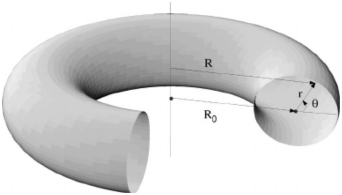
Figure : [McGuire et al., PRL, 1983]

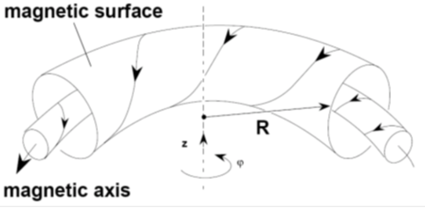
Trapped particle trajectories (passing are confined too!)



- $\omega_c \gg \omega_{bounce} > \omega_{precession} \longrightarrow$ possible decoupling of the three dynamics: Full kinetic, Gyrokinetic or Bounce-averaged descriptions
- For elegant derivation, using Lie algebra, see [Littlejohn, PS, 1982]
  $(\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{X_{gc}}, v_\parallel, (J_c, \zeta_c)) \xrightarrow{\omega \ll \omega_c} (\mathbf{X_{gc}}, v_\parallel; J_c)$
  $(\mathbf{X_{gc}}, v_\parallel; J_c) \rightarrow (\alpha, \beta, (J_\parallel, \zeta_\parallel); J_c) \xrightarrow{\omega \ll \omega_b} (\alpha, \beta; J_\parallel, J_c) \rightarrow ((J_p, \zeta_p); J_\parallel, J_c))$

The small "inverse aspect ratio" Tokamak





- $\epsilon = a/R_0 \ll 1 \quad \longrightarrow \quad \mathbf{B} = B_T\,\hat{\varphi} + \nabla\psi \times \nabla\varphi$
  $B_T = B_0 \frac{R_0}{R} \simeq B_0 \left(1 - \frac{r}{R_0}\cos\theta\right) \quad \& \quad \psi = RA_\varphi \simeq R_0 A_\varphi(r, \theta, \varphi, t)$
  with $\psi_{eq} = \psi_{eq}(r)$ (concentric circular cross sections at equilibrium)
- Safety factor: $q(r) = \frac{rB_{\theta,eq}}{R_0 B_T}$ where $B_{\theta,eq} = -R_0^{-1}\partial_r\psi_{eq} = -R_0^{-1}\psi'_{eq}$
  i.e. magnetic winding number
- Note: $B_\theta \ll B_T$ & $q = O(1)$

Equilibrium trajectories: bounce motion

- Let's take $(\partial_r)^{-1} \gg \rho_c \longrightarrow J_c \simeq \mu = \frac{1}{2} m v_{gyro}^2 / B_T \simeq cst$

$$\mathcal{E} = \frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 \simeq \underbrace{\frac{1}{2} m v_\parallel^2}_{\mathcal{E}_{kin}\ \text{“along“ field line}} + \underbrace{\mu B_T(r,\theta)}_{potential\ \ well,\ r \simeq cst}$$

  - Circular cross section $\longrightarrow$ A classical pendulum :-))
    $v_\parallel^2 = (r^2 + R_0^2 q^2(r))\dot\theta^2 \simeq R_0^2 q^2(r)\dot\theta^2 \quad \& \quad B_T = B_0 \left(1 - \frac{r}{R_0}\cos\theta\right)$

$$\frac{\mathcal{E}}{mR_0^2 q^2} = \frac{1}{2}\dot\theta^2 - \underbrace{\frac{\mu B_0 r}{mR_0^3 q^2}}_{=\ g/l}\cos\theta + \underbrace{\frac{\mu B_0}{mR_0^2 q^2}}_{=cst}$$

  - Trapping parameter: $k^2 = \frac{e+1}{2}, \quad e = \frac{\mathcal{E} - \mu B_0}{\mu B_0}\frac{R_0}{r}$ ($k^2 \in [0,1]$ for trapped particles)

$$J_\parallel = \frac{1}{2\pi}\oint m v_\parallel ds_\parallel = \frac{8}{\pi} R_0^{-1/2} q \sqrt{m\mu B_0}\left[(k^2 - 1)K(k) + E(k)\right]$$

  - Bounce period: $T_b = 2\pi\omega_b^{-1} = 2\pi\,\partial J_\parallel / \partial\mathcal{E} = 4R_0 q \sqrt{\frac{mR_0}{\mu B_0}} K(k)$

Banana width & precessional motion

- $P_\varphi = mRv_\varphi + \frac{e}{c}\psi$ is an exact invariant (φ is cyclic at the equilibrium)
  $v_\varphi \simeq v_\parallel$ changes along the trajectory $\longrightarrow$ ψ (i.e. r) must change too
- $P_\varphi = mR_0 v_\parallel|_{\theta=0} + \frac{e}{c}\psi_{eq}(r) + \frac{e}{c}\psi'_{eq}\delta r|_{\theta=0} = P_\varphi|_{turning\ \ point} = \frac{e}{c}\psi_{eq}(r)$

$\longrightarrow$ Banana orbit width: $\delta r|_{\theta=0} = \frac{qR_0}{\omega_c r} v_\parallel|_{\theta=0}$

- Two possible reasons for toroidal precession:
  1) Magnetic shear: forward/backward motions are not along the same line

$$\Delta\varphi = \oint d\varphi = \underbrace{\oint q d\theta}_{=0} + \oint q'\delta r d\theta \simeq q'(r)\delta r|_{\theta=0} \;\rightarrow\; \omega_{D,1} \simeq \frac{q'(r)}{m\omega_c r} J_\parallel \omega_b$$

  2) The magnitude of $v_\parallel$ is not the same during forward/backward motion

$$\delta\mathcal{E}_{k,\parallel}|_{\theta=0} = \mu B_0 \delta r|_{\theta=0} \longrightarrow \omega_{D,2} \simeq \frac{\delta v_\parallel|_{\theta=0}}{R_0} = \frac{\mu B_0 q}{m\omega_c R_0 r}$$

- It is possible to obtain ω_D rigorously ($\omega_D = \omega_{D,1} + \omega_{D,2}$) using

$$\frac{\omega_D}{\omega_b} = \frac{\partial\mathcal{E}}{\partial J_p}\frac{\partial J_\parallel}{\partial\mathcal{E}} = \frac{\partial J_\parallel}{\partial J_p} \; ; \; J_p = \oint P_\varphi d\varphi = P_\varphi = \frac{e}{c}\psi_{eq}(r) \;\rightarrow\; \omega_D = \frac{c}{e}\frac{1}{\psi'_{eq}}\frac{\partial J_\parallel}{\partial r}$$

Deeply trapped particles

- Let's take $v_{\parallel} \to 0$, thus:
  - $J_{\parallel} = 0$ and stays zero ($\omega \ll \omega_b$), as well as $v_{\parallel}$
  - μ and $J_{\parallel} = 0$ are parameters
  - (φ, P_φ) are the natural canonical variables: 2D phase space :-)
  - $P_\varphi \simeq \frac{e}{c}\psi$, i.e. $P_\varphi \leftrightarrow r$ & $\partial_{P_\varphi} \leftrightarrow \partial_r$
- The Hamiltonian $H = \frac{1}{2m}\left(\mathbf{P}_{\parallel} - \frac{e}{c}\mathbf{A}_{\parallel}\right)^2 + \mu B_T + e\phi$ reduces to

$$H(\varphi, P_\varphi) = \underbrace{\mu B_T(P_\varphi)}_{equilibrium} + \underbrace{e\phi(\varphi, P_\varphi, t)}_{mode}$$

$\Rightarrow$ The coupling is done only via the electric potential ϕ

$$\dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \frac{\partial \mu B_T}{\partial P_\varphi} + \frac{\partial e\phi}{\partial P_\varphi} = \underbrace{\mu \frac{\partial r}{\partial P_\varphi}\frac{\partial B_T(r)}{\partial r}}_{\omega_D = \frac{\mu B_0 q}{m\omega_c R_0 r}} + e\frac{\partial r}{\partial P_\varphi}\frac{\partial \phi(r,\varphi,t)}{\partial r}$$

$$\dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} = e\frac{\partial \phi(r,\varphi,t)}{\partial \varphi}$$

A reduced Fishbone model: Fast particle response

- Take into account only deeply trapped particles ($v_{\|} = 0$) with a single value for the magnetic moment, $\mu = \mu_*$

  $\Rightarrow$ Reduction of the phase space from 6D to 2D (φ, P_φ)
- All fast particles are contained well inside the $q = 1$ surface ("core region")
- The dominant mode has a kink-like shape (mode numbers $m = n = 1$, electric potential $\phi/r \simeq cst$ well inside of the $q = 1$ surface)

$$\frac{\partial f}{\partial t} + [H, f] = 0 \quad ; \quad H(\varphi, P_\varphi) = \mu_* B_T(P_\varphi) + e\phi(\varphi, P_\varphi)$$

$$\frac{\partial \mu_* B_T}{\partial P_\alpha} = \omega_D(P_\alpha) = \frac{\mu B_T q(P_\varphi)}{\omega_C m R r(P_\varphi)} \quad ; \quad \phi = r(P_\varphi)\frac{\phi_0(t)}{r_0} e^{i\varphi}$$

- Note: particles interact with a single mode. All the other ones vanish far quicker in the core region where particles are.

A reduced Fishbone model: Bulk plasma response

- Fluid description for the bulk of the plasma, neglecting the thermal pressure effects and density variations $\longrightarrow$ Reduced-MHD description.

$$\frac{\partial \psi}{\partial t} + \{\phi, \psi\} = 0$$

$$\frac{\partial \Delta\phi}{\partial t} + \{\phi, \Delta\phi\} - \{\psi, \Delta\psi\} = \widetilde{\rho} = \alpha_{normalization} \left[(\hat{\varphi} \times \kappa) \cdot \nabla P_{\perp,h} \right]$$

- Toroidal effects are retained only for the contribution given by fast particles (κ is the toroidal curvature and $P_{\perp,h}$ the fast particle pressure) $\longrightarrow$ Cylindrical geometry
- As before we set $\phi = r \frac{\phi_0(t)}{r_0} e^{i\varphi - i\theta}$ well inside the $q = 1$ surface
- On the contrary strong variations are allowed across the $q = 1$ surface and finally $\phi \to 0$ for $r \to a$
- One mode evolution: only Fast particle (kinetic) nonlinearities are retained, MHD nonlinearities are neglected

Linear theory: Analytic results I

- Let's take $f = F_{eq} + \delta f$, with $\delta f \ll F_{eq}$, and $\partial_t \to -i\omega$. The mode equation reads

$$-\omega^2 \left(\frac{\phi}{r}\right)' + \frac{v_{A,T}^2}{R_0^2}\left(1 - \frac{1}{q(r)}\right)^2 \left(\frac{\phi}{r}\right)' = -i\omega \frac{1}{r^3}\int_0^r d\bar{r}\bar{r}^2 \tilde{\rho}(\delta f)$$

$$\text{where} \quad \delta f = \frac{er\frac{\phi_0}{r_0}\frac{dF_{eq}}{dr}\frac{dr}{dP_\varphi}}{\omega_D(r) - \omega}$$

- Note as a spatial gradient corresponds to the usual velocity gradient. Here a "decreasing density" is equivalent to a bump on the tail.
- Finally a general dispersion relation is obtained :

$$i = K\int_0^1 \frac{y^2 q(y)^2 \left(-\frac{dF_{eq}}{dy}\right)}{q(y) - \frac{y\omega}{\omega_D(y=1)}} dy \qquad (1)$$

where K takes into account the energetic content for the fast particles, the MHD and geometric parameters and $y = r/r_*$ (r_* being so that $q(r_*) = 1$).

Linear theory: analytic results II

- Let's take: $F_{eq} = \frac{n_0}{2}\delta(\mu - \mu_*)\delta v_{||}(1 - \mathrm{erf}(\beta(y - y_0))$
$\longrightarrow$ analytic values the threshold condition and mode frequency at the threshold
$$K_0 = \frac{-1}{\sqrt{\pi}\beta y_0^3} \ ; \quad \omega_0 = \omega_D(y_0)\left(1 - \frac{1}{\beta^2 y_0^2}\right)$$
where y_0 is the position of the highest radial gradient in the distribution function
- Close to the threshold
$$K = K_0 + \delta K \quad ; \quad \omega = \omega_0 + \delta\omega + i\gamma$$
we obtain a growth rate and a correction for the real frequency:
$$\gamma \simeq \frac{\delta K}{K_0}\frac{\sqrt{\pi}}{2\beta y_0}\omega_0 \ ; \quad \delta\omega \simeq \frac{1}{2}\frac{\sqrt{\pi}}{\beta y_0}\gamma$$

Nonlinear numerical code

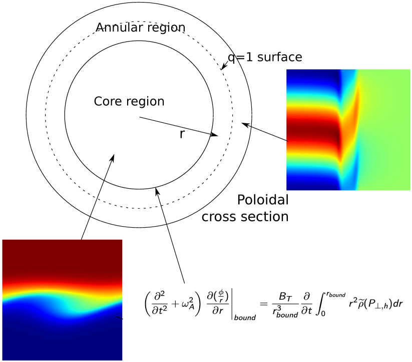
Based on the same domain decomposition:

1) The core region

- Nonlinear kinetic description:
  Semi-lagrangian code assuming
  $\phi = r\left(\frac{\phi}{r}\right)\Big|_{bound}$
- Linearized MHD response, including the fast particle pressure:
  i.e. an evolution equation for $\partial_r\left(\frac{\phi}{r}\right)\Big|_{bound}$
  plus the frozen-in equation for $\psi|_{bound}$

2) Thin annular region around $q = 1$ surface

- No fast particle here
- Cylindrical $\rightarrow$ slab description
- Semi-spectral MHD code
  (only one mode at present time)
- Uses $\partial_r\left(\frac{\phi}{r}\right)\Big|_{bound}$ & $\psi|_{bound}$ as B.C.
- Provides ψ & ϕ,
  in particular $\phi|_{bound}$ (the Hamiltonian)



Linear benchmark

With numerical simulations, the linear results are recovered

- Good agreement with frequency value
- Good agreement with growth rate and mode shape

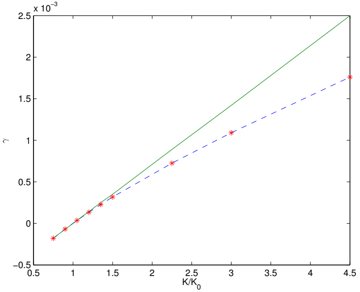


Figure : Growth rate as a function of K/K_0. Stars are the numerical values, the green line is the analytic prediction.

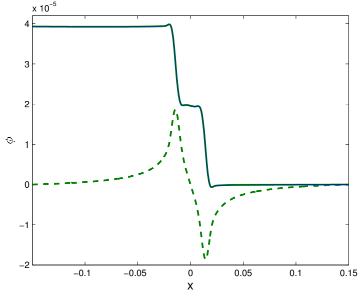


Figure : ϕ profile in the annular layer. In blue the real part, in red the imaginary part.

Nonlinear results: Mode saturation level

- The first local maximum of the amplitude is proportional to γ^2 i.e when the phase-space island width $\propto \sqrt{\phi}$ reaches the resonance width $\propto \gamma$ [Zonca et al., NJP, 2015]
- Amplitude oscillations are far larger compared to the usual "Bump on Tail" case, with [Berk et al., PLA, 1997] or without dissipation [O'Neil, PF, 1965]

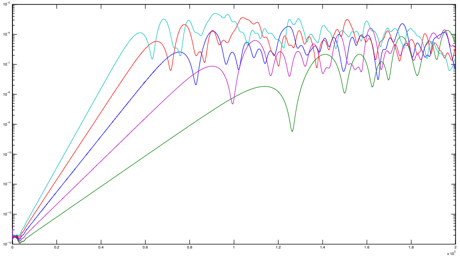


Figure : Evolution of the kinetic energy of the mode versus time, in logscale

Frequency chirping and particle ejection

Chirping is observed during the saturated phase (case studied here: $K/K_0 = 1.2$, giving $\gamma/\omega = 2.8\%$)

- Phase space structure motion matches frequency change
- Asymmetric system $\rightarrow$ Higher amplitude for the down chirping mode, i.e. particle ejection
- Outgoing particles continue to interact with the mode but actually are not trapped into the mode well

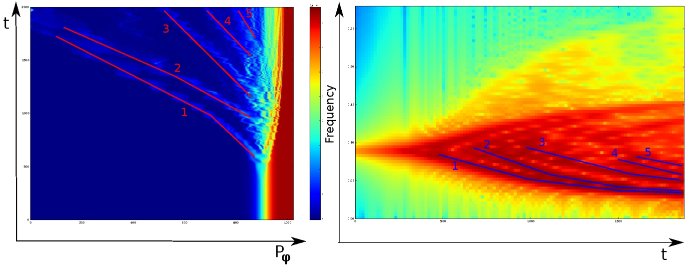


Figure : On the left : Evolution of the distribution function averaged over angle. On the right : spectrogram of the mode.

Structures in phase-space

- Dynamics close to the first maximum and minimum of the mode amplitude
- Only partial folding of f inside the phase-space island
- Strong stretching in φ & P_φ directions

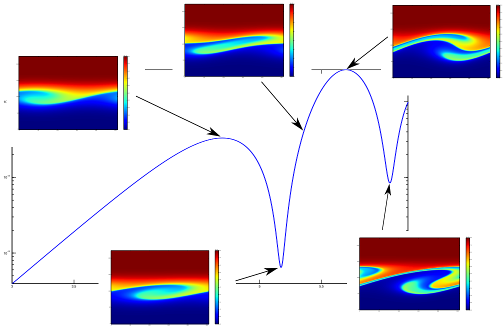
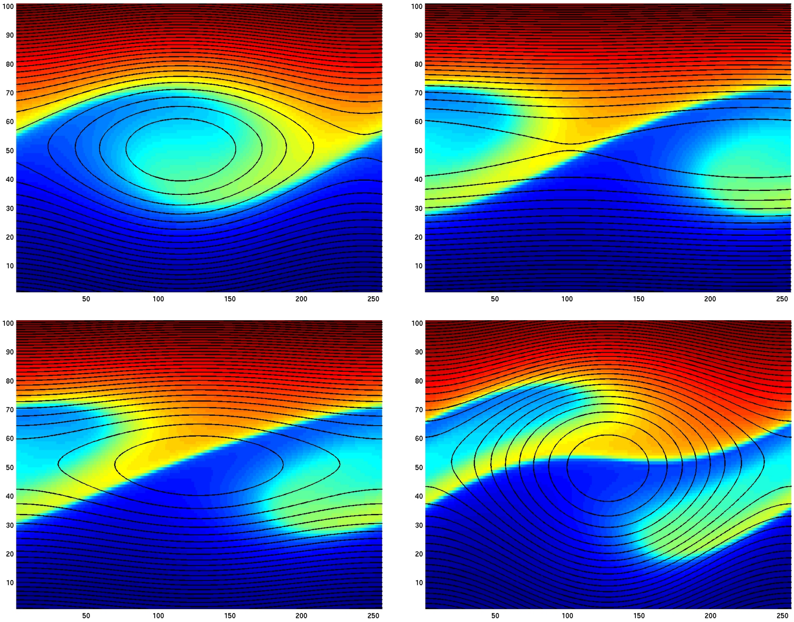


Figure : Distribution function at different stages of the saturation.

Island contraction and slippage



Contraction and slippage: The Fishbone peculiarity

The dispersion relation of the Fishbone is really different from the usual Bump on Tail one:

- BoT: marginally stable plasma wave + particle driver leading to instability i.e. $\Re D(\omega, k) \simeq 0$ & a small $\Im D$ gives the instability
  Thus when the energetic driver drops, the mode response is not dramatic.
- Fishbone exists only because fast particles are there, indeed $\omega_R = \omega_{D,h}$
  It is a genuine "energetic particle mode" [Zonca et al., NJP, 2015]
- The mode response to the energetic driver variation can be strong:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\phi}{r}\right)' + \omega_A^2(r)\left(\frac{\phi}{r}\right)' = -i\frac{\partial}{\partial t}\frac{1}{r^3}\int_0^r d\bar{r}\bar{r}^2\tilde{\rho}(f) = RHS$$

In the core $\omega^2_{A,bound} \gg \partial_{tt} \simeq \omega_R^2$ thus the mode is slave, compared to the particle driver:

$$\left(\frac{\phi}{r}\right)'\bigg|_{bound} \simeq \frac{RHS}{\omega_A^2|_{bound}} \quad \text{giving} \quad \phi|_{bound} = \phi_0 \propto RHS$$

Amplitude and phase simply follow: the mode is slave to the energetic driver.

Conclusions

- The Precessional Fishbone instability can be described by an hybrid Fluid-Hamiltonian model where the phase-space coordinates are quite unusual.
- This reduced model is able to catch the qualitative dynamics of the mode: the frequency downchirping of the mode and the gradual ejection of fast particles.
- It has the great advantage, over more complete models, of permitting an easier analysis of the structure dynamics in phase-space.
- Future perspectives:
  1) Allowing MHD nonlinearities to develops around the $q = 1$ surface.
  2) Looking at a more general approach for "energetic particle modes".