Nonlinear hybrid simulations of precessional Fishbone instability

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The Fishbone Instability

- First observation on the PDX tokamak, with near perpendicular neutral beam injection active
- Bursts of electromagnetic instability, associated with losses of fast particles
- Mode frequency consistent with trapped fast ion precession frequency
- Dominant $m = n = 1$, kink-shaped, mode structure
- Observation of frequency chirping

Figure: [McGuire et al., PRL, 1983]
Trapped particle trajectories (passing are confined too!)

- \( \omega_c \gg \omega_{bounce} > \omega_{precession} \rightarrow \) possible decoupling of the three dynamics: Full kinetic, Gyrokinetic or Bounce-averaged descriptions

- For elegant derivation, using Lie algebra, see [Littlejohn, PS, 1982]

\[
\begin{align*}
(x, v) & \rightarrow (X_{gc}, v_\parallel, (J_c, \zeta_c)) \\
& \xrightarrow{\omega \ll \omega_c} (X_{gc}, v_\parallel; J_c) \\
(X_{gc}, v_\parallel; J_c) & \rightarrow (\alpha, \beta, (J_\parallel, \zeta_\parallel); J_c) \\
& \xrightarrow{\omega \ll \omega_b} (\alpha, \beta; J_\parallel, J_c) \rightarrow ((J_p, \zeta_p); J_\parallel, J_c)
\end{align*}
\]
The small "inverse aspect ratio" Tokamak

\[ \epsilon = \frac{a}{R_0} \ll 1 \rightarrow \mathbf{B} = B_T \hat{\phi} + \nabla \psi \times \nabla \varphi \]

\[ B_T = B_0 \frac{R_0}{R} \approx B_0 \left(1 - \frac{r}{R_0} \cos \theta \right) \quad \& \quad \psi = RA_\varphi \approx R_0 A_\varphi (r, \theta, \varphi, t) \]

with \( \psi_{eq} = \psi_{eq} (r) \) (concentric circular cross sections at equilibrium)

\[ q(r) = \frac{r B_{\theta, eq}}{R_0 B_T} \]

where \( B_{\theta, eq} = -R_0^{-1} \partial_r \psi_{eq} = -R_0^{-1} \psi'_{eq} \)

i.e. magnetic winding number

\[ B_\theta \ll B_T \quad \& \quad q = O(1) \]
Equilibrium trajectories: bounce motion

- Let’s take $(\partial_r)^{-1} \gg \rho_c \implies J_c \simeq \mu = \frac{1}{2} mv_{\text{gyro}}^2 / B_T \simeq \text{cst}$

$$\mathcal{E} = \frac{1}{2} mv_\parallel^2 + \frac{1}{2} mv_\perp^2 \simeq \frac{1}{2} mv_\parallel^2 + \mu B_T(r, \theta)$$

$\mathcal{E}_\text{kin} \text{ "along" field line}$

- Potential well, $r \simeq \text{cst}$

- Circular cross section $\implies$ A classical pendulum :-))

$$v^2_\parallel = (r^2 + R_0^2 q^2(r))\dot{\theta}^2 \simeq R_0^2 q^2(r)\dot{\theta}^2 \quad \& \quad B_T = B_0 \left(1 - \frac{r}{R_0} \cos \theta\right)$$

$$\frac{\mathcal{E}}{m R_0^2 q^2} = \frac{1}{2} \dot{\theta}^2 - \frac{\mu B_0 r}{m R_0^3 q^2} \cos \theta + \frac{\mu B_0}{m R_0^2 q^2} = \frac{g}{l} = \text{cst}$$

- Trapping parameter: $k^2 = \frac{e+1}{2}, \quad e = \frac{\mathcal{E} - \mu B_0 R_0}{\mu B_0 r} \quad (k^2 \in [0, 1] \text{ for trapped particles})$

$$J_\parallel = \frac{1}{2\pi} \int m v_\parallel ds_\parallel = \frac{8}{\pi} R_0^{-1/2} q \sqrt{m \mu B_0} \left[(k^2 - 1)K(k) + E(k)\right]$$

- Bounce period: $T_b = 2\pi \omega_b^{-1} = 2\pi \partial J_\parallel / \partial \mathcal{E} = 4R_0 q \sqrt{\frac{m R_0}{\mu B_0}} K(k)$
Banana width & precessional motion

- \( P_\varphi = mRv_\varphi + \frac{e}{c}\psi \) is an exact invariant (\( \varphi \) is cyclic at the equilibrium)
  
  \( v_\varphi \simeq v_\parallel \) changes along the trajectory \( \rightarrow \) \( \psi \) (i.e. \( r \)) must change too

- \( P_\varphi = mR_0v_\parallel|_{\theta=0} + \frac{e}{c}\psi_{eq}(r) + \frac{e}{c}\psi_{eq}'\delta r|_{\theta=0} = P_\varphi|_{\text{turning point}} = \frac{e}{c}\psi_{eq}(r) \)

  \( \rightarrow \) Banana orbit width: \( \delta r|_{\theta=0} = \frac{qR_0}{\omega_cr} v_\parallel|_{\theta=0} \)

- Two possible reasons for toroidal precession:
  1) Magnetic shear: forward/backward motions are not along the same line

  \[
  \Delta \varphi = \int d\varphi = \int qd\theta + \int q'\delta r d\theta \simeq q'(r)\delta r|_{\theta=0} \rightarrow \omega_D,1 \simeq \frac{q'(r)}{m\omega_c r}J_\parallel\omega_b
  \]

  2) The magnitude of \( v_\parallel \) is not the same during forward/backward motion

  \[
  \delta E_{k,\parallel}|_{\theta=0} = \mu B_0\delta r|_{\theta=0} \rightarrow \omega_D,2 \simeq \frac{\delta v_\parallel|_{\theta=0}}{R_0} = \frac{\mu B_0q}{m\omega_cR_0r}
  \]

- It is possible to obtain \( \omega_D \) rigorously (\( \omega_D = \omega_D,1 + \omega_D,2 \)) using

  \[
  \frac{\omega_D}{\omega_b} = \frac{\partial E}{\partial J_\parallel} \frac{\partial J_\parallel}{\partial E} = \frac{\partial J_\parallel}{\partial J_p} ; J_p = \int P_\varphi d\varphi = P_\varphi = \frac{e}{c}\psi_{eq}(r) \rightarrow \omega_D = \frac{c}{e\psi_{eq}'(r)} \frac{\partial J_\parallel}{\partial r}
  \]
Deeply trapped particles

- Let's take \( v_\parallel \to 0 \), thus:
  - \( J_\parallel = 0 \) and stays zero \( (\omega \ll \omega_b) \), as well as \( v_\parallel \)
  - \( \mu \) and \( J_\parallel = 0 \) are parameters
  - \( (\varphi, P_\varphi) \) are the natural canonical variables: 2D phase space :-)
  - \( P_\varphi \simeq \frac{e}{c} \psi \), i.e. \( P_\varphi \leftrightarrow r \) & \( \partial P_\varphi \leftrightarrow \partial r \)

- The Hamiltonian \( H = \frac{1}{2m} (P_\parallel - \frac{e}{c} A_\parallel)^2 + \mu B_T + e\phi \) reduces to

\[
H(\varphi, P_\varphi) = \mu B_T(P_\varphi) + e\phi(\varphi, P_\varphi, t)
\]

\(
\text{equilibrium} \hspace{1cm} \text{mode}
\)

- The coupling is done only via the electric potential \( \phi \)

\[
\dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \mu B_T \frac{\partial r}{\partial P_\varphi} + e \frac{\partial \phi}{\partial P_\varphi}
\]

\[
\omega_D = \frac{\mu B_0 q}{m \omega_c R_0 r}
\]

\[
\dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} = e \frac{\partial \phi(r, \varphi, t)}{\partial \varphi}
\]
A reduced Fishbone model: Fast particle response

- Take into account only deeply trapped particles ($v_{\parallel} = 0$) with a single value for the magnetic moment, $\mu = \mu^\ast$.  
  \[ \Rightarrow \text{Reduction of the phase space from 6D to 2D} \left( \varphi, P_\varphi \right) \]

- All fast particles are contained well inside the $q = 1$ surface ("core region")

- The dominant mode has a kink-like shape (mode numbers $m = n = 1$, electric potential $\phi/r \simeq \text{cst}$ well inside of the $q = 1$ surface)

\[
\frac{\partial f}{\partial t} + [H, f] = 0 \quad ; \quad H(\varphi, P_\varphi) = \mu^\ast B_T(P_\varphi) + e\phi(\varphi, P_\varphi)
\]

\[
\frac{\partial \mu^\ast B_T}{\partial P_\alpha} = \omega_D(P_\alpha) = \frac{\mu B_T q(P_\varphi)}{\omega_C m R r(P_\varphi)} \quad ; \quad \phi = r(P_\varphi) \frac{\phi_0(t)}{r_0} e^{i\varphi}
\]

- Note: particles interact with a single mode. All the other ones vanish far quicker in the core region where particles are.
A reduced Fishbone model: Bulk plasma response

- Fluid description for the bulk of the plasma, neglecting the thermal pressure effects and density variations $\rightarrow$ Reduced-MHD description.

$$\frac{\partial \psi}{\partial t} + \{\phi, \psi\} = 0$$

$$\frac{\partial \Delta \phi}{\partial t} + \{\phi, \Delta \phi\} - \{\psi, \Delta \psi\} = \tilde{\rho} = \alpha_{\text{normalization}} [(\hat{\phi} \times \kappa) \cdot \nabla P_{\perp,h}]$$

- Toroidal effects are retained only for the contribution given by fast particles ($\kappa$ is the toroidal curvature and $P_{\perp,h}$ the fast particle pressure) $\rightarrow$ Cylindrical geometry

- As before we set $\phi = r \frac{\phi_0(t)}{r_0} e^{i\varphi - i\theta}$ well inside the $q = 1$ surface

- On the contrary strong variations are allowed across the $q = 1$ surface and finally $\phi \rightarrow 0$ for $r \rightarrow a$

- One mode evolution: only Fast particle (kinetic) nonlinearities are retained, MHD nonlinearities are neglected
Let's take $f = F_{eq} + \delta f$, with $\delta f \ll F_{eq}$, and $\partial_t \to -i\omega$. The mode equation reads

$$-\omega^2 \left( \frac{\phi}{r} \right)' + \frac{\nu_{A,T}^2}{R_0^2} \left( 1 - \frac{1}{q(r)} \right)^2 \left( \frac{\phi}{r} \right)' = -i\omega \frac{1}{r^3} \int_0^r d\tilde{r} \tilde{r}^2 \tilde{\rho}(\delta f)$$

where

$$\delta f = \frac{er \frac{\phi_0}{r_0} \frac{dF_{eq}}{dr} \frac{dr}{dP_\phi}}{\omega_D(r) - \omega}$$

Note as a spatial gradient corresponds to the usual velocity gradient. Here a "decreasing density" is equivalent to a bump on the tail.

Finally a general dispersion relation is obtained:

$$i = K \int_0^1 \frac{y^2 q(y)^2 \left( -\frac{dF_{eq}}{dy} \right)}{q(y) - \frac{y\omega}{\omega_D(y=1)}} dy$$

(1)

where $K$ takes into account the energetic content for the fast particles, the MHD and geometric parameters and $y = r/r_*$ ($r_*$ being so that $q(r_*) = 1$).
Let's take: $F_{eq} = \frac{n_0}{2} \delta(\mu - \mu_*) \delta v_\parallel (1 - \text{erf}(\beta(y - y_0)))$

$\rightarrow$ analytic values the threshold condition and mode frequency at the threshold

$$K_0 = \frac{-1}{\sqrt{\pi} \beta y_0^3} ; \quad \omega_0 = \omega_D(y_0) \left(1 - \frac{1}{\beta^2 y_0^2}\right)$$

where $y_0$ is the position of the highest radial gradient in the distribution function

Close to the threshold

$$K = K_0 + \delta K ; \quad \omega = \omega_0 + \delta \omega + i \gamma$$

we obtain a growth rate and a correction for the real frequency:

$$\gamma \approx \frac{\delta K}{K_0} \frac{\sqrt{\pi}}{2\beta y_0} \omega_0 ; \quad \delta \omega \approx \frac{1}{2} \frac{\sqrt{\pi}}{\beta y_0} \gamma$$
Nonlinear numerical code

Based on the same domain decomposition:

1) The core region
   ▶ Nonlinear kinetic description:
     Semi-lagrangian code assuming
     \( \phi = r \left( \frac{\phi}{r} \right) \bigg|_{\text{bound}} \)
   ▶ Linearized MHD response, including the fast particle pressure:
     i.e. an evolution equation for \( \partial_r \left( \frac{\phi}{r} \right) \bigg|_{\text{bound}} \)
     plus the frozen-in equation for \( \psi \bigg|_{\text{bound}} \)

2) Thin annular region around \( q = 1 \) surface
   ▶ No fast particle here
   ▶ Cylindrical \( \rightarrow \) slab description
   ▶ Semi-spectral MHD code
     (only one mode at present time)
   ▶ Uses \( \partial_r \left( \frac{\phi}{r} \right) \bigg|_{\text{bound}} \) & \( \psi \bigg|_{\text{bound}} \) as B.C.
   ▶ Provides \( \psi \) & \( \phi \),
     in particular \( \phi \bigg|_{\text{bound}} \) (the Hamiltonian)
Linear benchmark

With numerical simulations, the linear results are recovered

- Good agreement with frequency value
- Good agreement with growth rate and mode shape

Figure: Growth rate as a function of $K/K_0$. Stars are the numerical values, the green line is the analytic prediction.

Figure: $\phi$ profile in the annular layer. In blue the real part, in red the imaginary part.
Nonlinear results: Mode saturation level

- The first local maximum of the amplitude is proportional to $\gamma^2$ i.e when the phase-space island width $\propto \sqrt{\phi}$ reaches the resonance width $\propto \gamma$ [Zonca et al., NJP, 2015]
- Amplitude oscillations are far larger compared to the usual “Bump on Tail” case, with [Berk et al., PLA, 1997] or without dissipation [O’Neil, PF, 1965]

Figure: Evolution of the kinetic energy of the mode versus time, in logscale
Frequency chirping and particle ejection

Chirping is observed during the saturated phase (case studied here: $K/K_0 = 1.2$, giving $\gamma/\omega = 2.8\%$)

- Phase space structure motion matches frequency change
- Asymmetric system $\rightarrow$ Higher amplitude for the down chirping mode, i.e. particle ejection
- Outgoing particles continue to interact with the mode but actually are not trapped into the mode well

Figure: On the left: Evolution of the distribution function averaged over angle. On the right: spectrogram of the mode.
Structures in phase-space

- Dynamics close to the first maximum and minimum of the mode amplitude
- Only partial folding of $f$ inside the phase-space island
- Strong stretching in $\varphi$ & $P_\varphi$ directions

Figure: Distribution function at different stages of the saturation.
Island contraction and slippage
Contraction and slippage: The Fishbone peculiarity

The dispersion relation of the Fishbone is really different from the usual Bump on Tail one:

- BoT: marginally stable plasma wave + particle driver leading to instability
  i.e. $\Re D(\omega, k) \approx 0$ & a small $\Im D$ gives the instability
  Thus when the energetic driver drops, the mode response is not dramatic.
- Fishbone exists only because fast particles are there, indeed $\omega_R = \omega_{D,h}$
  It is a genuine “energetic particle mode” [Zonca et al., NJP, 2015]
- The mode response to the energetic driver variation can be strong:

$$\frac{\partial^2}{\partial t^2} \left( \frac{\phi}{r} \right)' + \omega_A^2(r) \left( \frac{\phi}{r} \right)' = -i \frac{\partial}{\partial t} \frac{1}{r^3} \int_0^r d\bar{r} \bar{r}^2 \tilde{\rho}(f) = \text{RHS}$$

In the core $\omega_{A,\text{bound}} \gg \partial_{tt} \sim \omega_R^2$ thus the mode is slave, compared to the particle driver:

$$\left( \frac{\phi}{r} \right)' \bigg|_{\text{bound}} \sim \frac{\text{RHS}}{\omega_A^2|_{\text{bound}}} \quad \text{giving} \quad \phi|_{\text{bound}} = \phi_0 \propto \text{RHS}$$

Amplitude and phase simply follow: the mode is slave to the energetic driver.
Conclusions

- The Precessional Fishbone instability can be described by an hybrid Fluid-Hamiltonian model where the phase-space coordinates are quite unusual.
- This reduced model is able to catch the qualitative dynamics of the mode: the frequency downchirping of the mode and the gradual ejection of fast particles.
- It has the great advantage, over more complete models, of permitting an easier analysis of the structure dynamics in phase-space.
- Future perspectives:
  1) Allowing MHD nonlinearities to develops around the $q = 1$ surface.
  2) Looking at a more general approach for “energetic particle modes”.