A quasi-neutral kinetic model for collisionless plasma

Enrico Camporeale

Multiscale Dynamics
Center for Mathematics and Computer Science (CWI)
Amsterdam, Netherlands

In collaboration with C. Tronci (U. Surrey, UK), F. Deluzet (U. Toulouse, France), D. Burgess (Queen Mary U. London)

CWI
Centrum Wiskunde & Informatica

VLASOVIA 2016
Copanello, 30th May 2016

Motivation and goals

- A self-consistent description of the coupling between ion and electron scales (in space and time) remains the holy grail of computational plasma physics;
- It makes sense to use Reduced models (hybrid, gyro, etc.) only if we can critically assess their range of validity;
- Finally, can we devise a model that is computationally cheaper than full Vlasov-Maxwell, but still accurate enough to be used for sub-ion scales?

A quick reminder: what is wrong with Vlasov-Maxwell?

- It is the 'most complete' description of collisionless plasma dynamics
- Of course, the computational cost depends on the numerical implementation (discretization in time and space)
- Explicit schemes are typically constrained by resolving:
  - Debye length in space
  - Electron plasma frequency in time
  - Courant–Friedrichs–Lewy (CFL) condition → Dx/Dt < c

  CFL is very often the most stringent!

Numerology of Solar Wind simulations

Typical solar wind parameters:

$T = 10$ eV, $n = 10$ cm^{-3} , $B = 6$ nT, $\lambda_d \sim 7$ m, $f_{pe} \sim 30$ kHz

$\rho_e / \lambda_d \sim 170 \rightarrow \rho_i / \lambda_d \sim \mathrm{Sqrt}(m_i/m_e) * 170 \sim 7200$

$\Delta x < \pi \lambda_d \rightarrow$ 1 ion gyroradius needs 2300 cells per dimension

$\omega_{pe} / \Omega_{ce} \sim 170 \rightarrow \omega_{pe} / \Omega_{ci} \sim (m_i/m_e) * 170 = 310{,}000$

$\Delta t\, \omega_{pe} = 1 \rightarrow$ 1 ion gyroperiod needs 2 millions timesteps

$\rightarrow$ 1 electron gyroperiod needs 1000 timesteps

$c\, \Delta t/\Delta x \sim 430 \rightarrow$ CFL condition not satisfied!

A realistic fully-kinetic simulation of the solar wind is **practically impossible** with an explicit code, due to the large scale separation involved, both in time and space.

What are the alternatives?

- If high (close to plasma frequency) and low-frequency coupling must be retained, (semi-) implicit methods might be the only way
- Otherwise, reduced models:
  - Hybrid kinetic-fluid
  - Gyrokinetics
  - Other fluid models with kinetic closures (Landau fluid, etc.)

The 'hybrid' model in a snapshot

- Quasi-neutral (Debye length is much smaller than the characteristic spatial scales, and the characteristic frequencies are much smaller than the electron plasma frequency) → Displacement current is negligible

- No Gauss's law (Poisson equation)

- Ions are treated kinetically (solve Vlasov equation), electrons as neutralizing fluid

- It needs a closure for electrons (Equation of state)

- The electric field can be derived by Ohm's law:

$$\mathbf{E} - \frac{d_e^2}{n}\Delta\mathbf{E} = -(\mathbf{u}\times\mathbf{B}) + \frac{1}{n}(\mathbf{j}\times\mathbf{B}) + \frac{1}{n}d_e^2\nabla\cdot\boldsymbol{\Pi} - \frac{1}{n}\nabla P_e + \frac{d_e^2}{n}\nabla\cdot[\mathbf{uj}+\mathbf{ju}] - \frac{1}{n}d_e^2\nabla\cdot\left(\frac{\mathbf{jj}}{n}\right)$$

From Valentini et al. *J. Comp. Phys.* (2007)

- Magnetic field is advanced through Faraday's law

The gyrokinetic model in a snapshot

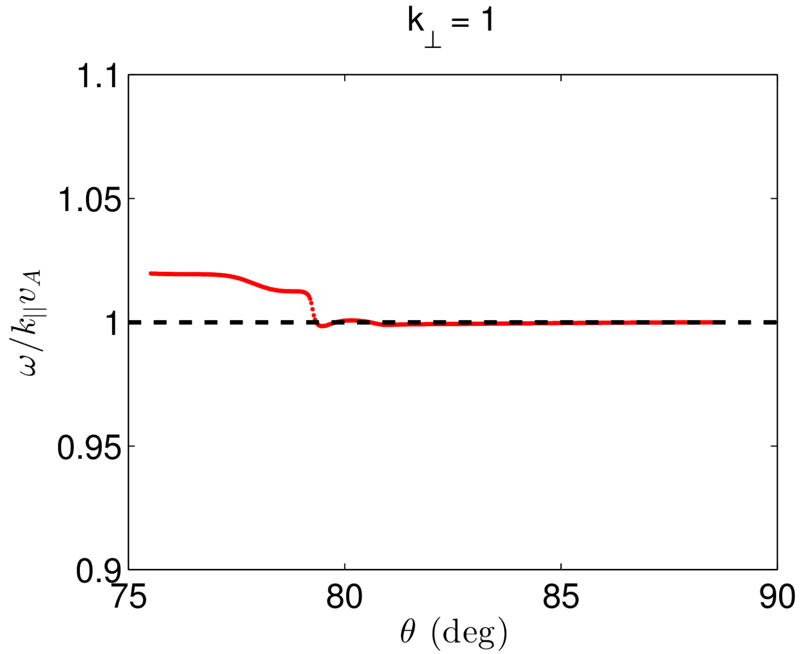
- It is a rigorous limit of full Vlasov;
- Quasi-neutral
- It requires $k_{||} << k_{\perp}$ and $\omega << \Omega_i$
- No cyclotron frequency and high frequency physics
- It reduces the computational complexity of Vlasov-Maxwell to 3D-2V (dimensionality reduction)

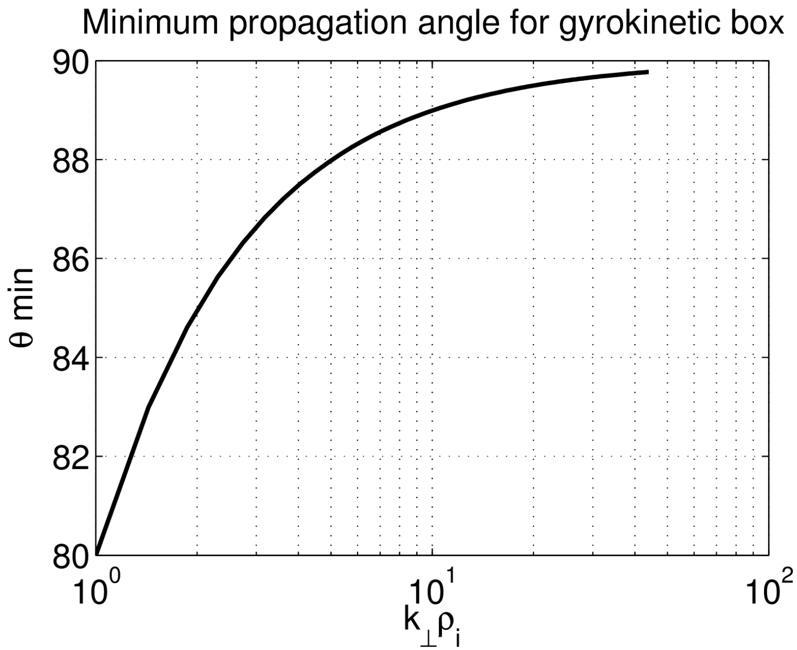
Linear theory case study

- Typical solar wind parameters taken from Salem et al. *ApJ* (2012):
  - B = 11 nT; Te = 13 eV; Tp = 13.6 eV; n = 9 cm^{-3}. Plasma beta = 0.4; $\omega_{pe} \sim 90\ \Omega_{ce}$
- We aim at assessing the range of validity of hybrid and gyrokinetic models in the range $k\rho_i$ = [0.1,44] by comparison with the full Vlasov-Maxwell model
- Obviously a good agreement in the linear regime is a minimal requirement and it does not guarantees accuracy in the non-linear regime!

A preliminary consideration on the geometry of computational box

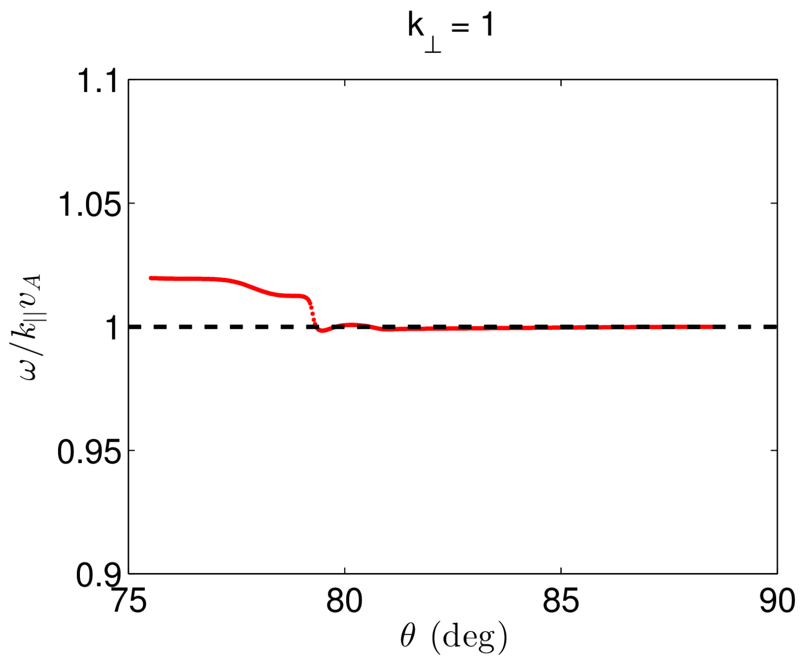
- Gyrokinetics: the requirement $k_{||} << k_{\perp}$ automatically defines $k_{||, max}$ → spatial resolution in parallel direction. This turns out to be a function of $k_{\perp, min}$ (the box length in perp direction), due to $\omega / k_{||} v_A = f(k_{\perp})$

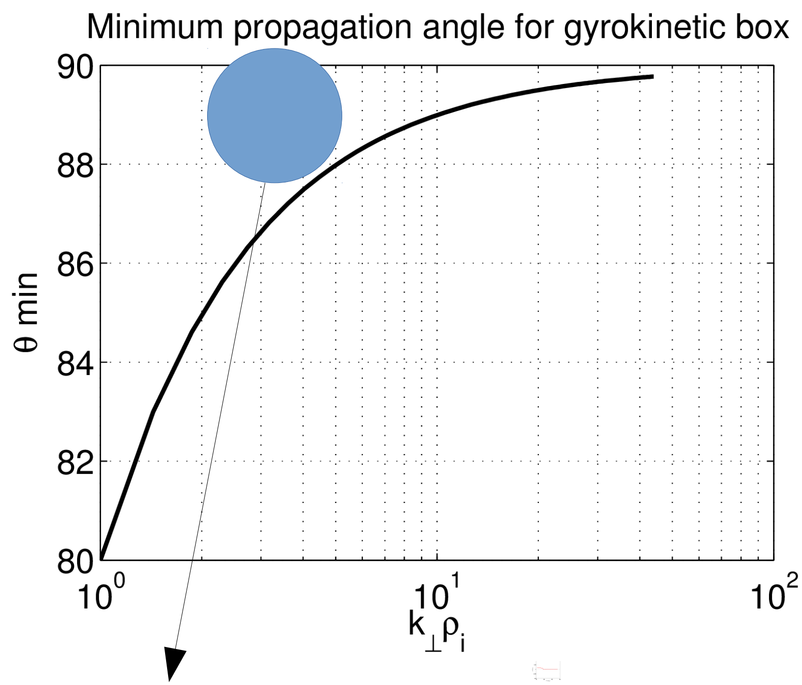




A preliminary consideration on the geometry of computational box

- Gyrokinetics: the requirement $k_{||} << k_{\perp}$ automatically defines $k_{||, max}$ → spatial resolution in parallel direction. This turns out to be a function of $k_{\perp, min}$ (the box length in perp direction), due to $\omega / k_{||} v_A = f(k_{\perp})$

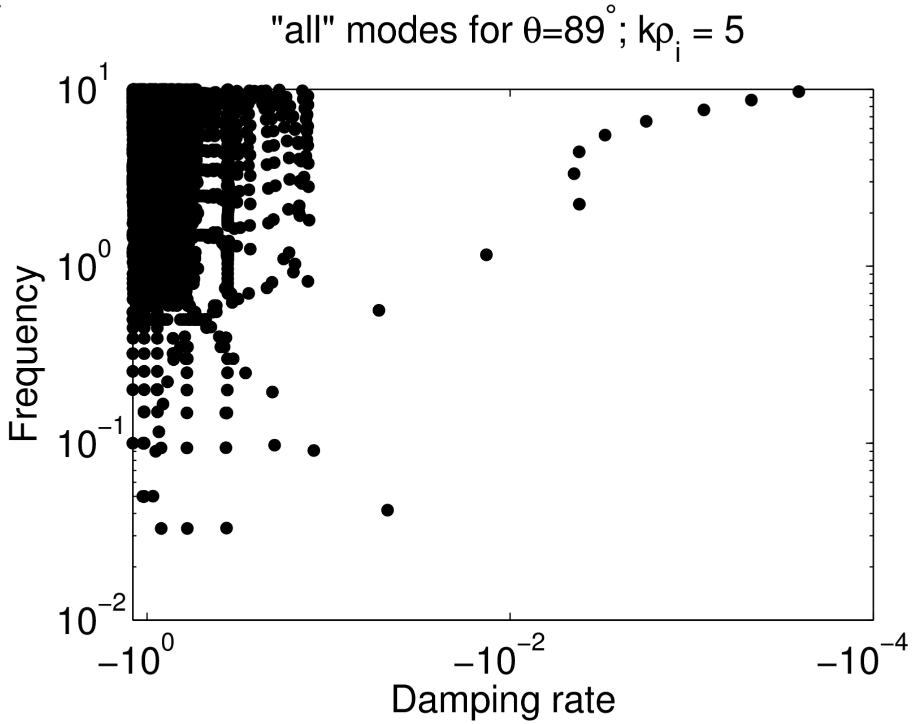




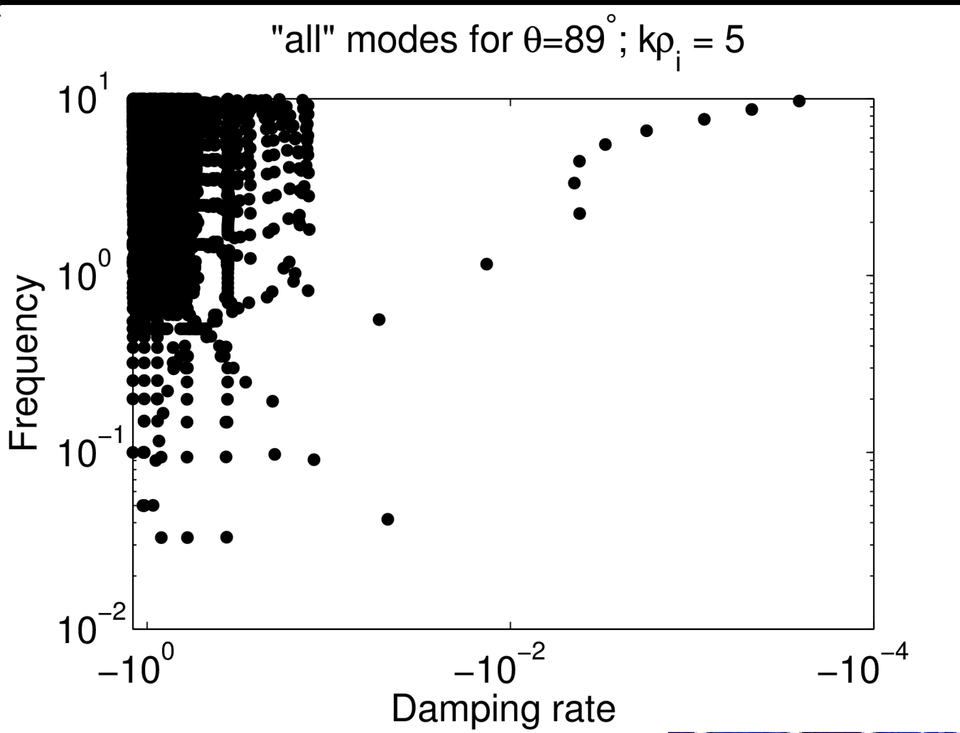
What are the linear modes in this region?

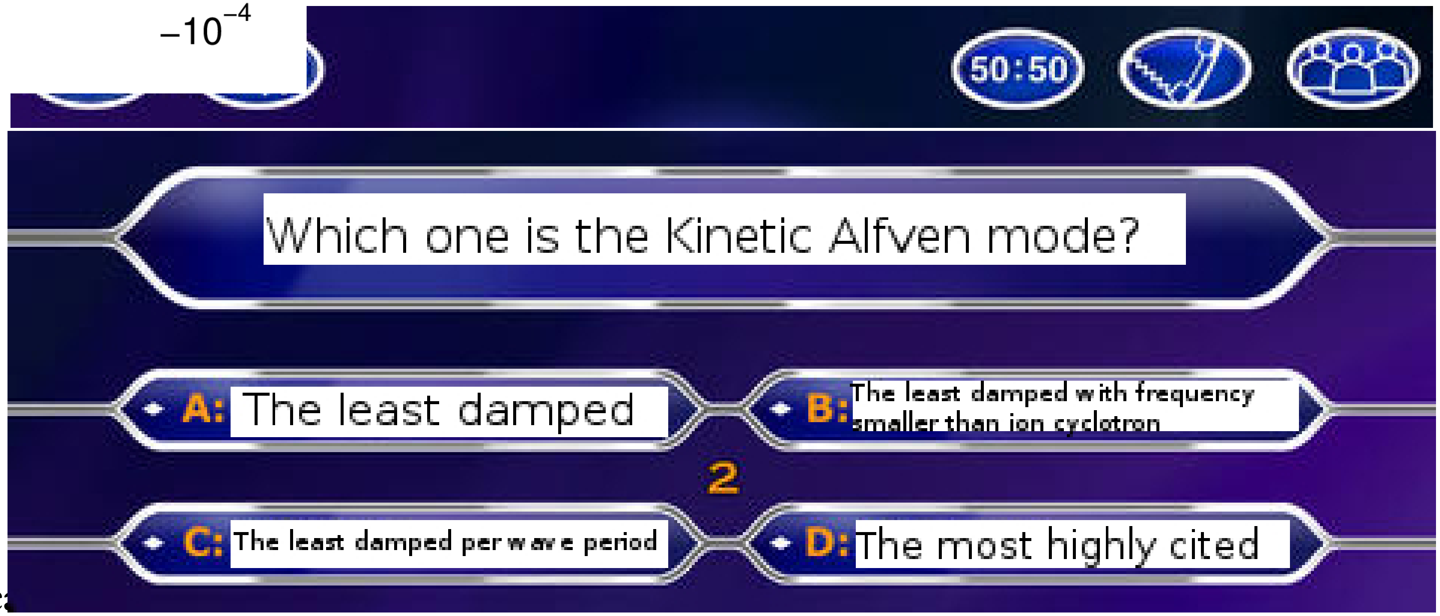
Enrico Camporeale – e.camporeale@cwi.nl

Why are Kinetic Alfven waves so special?

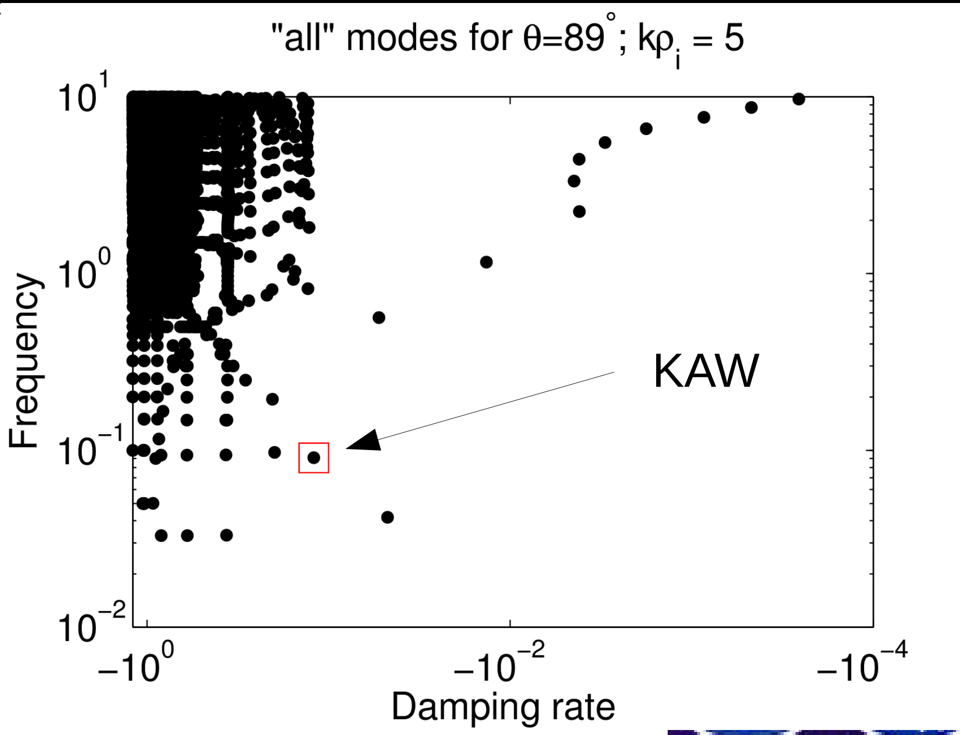


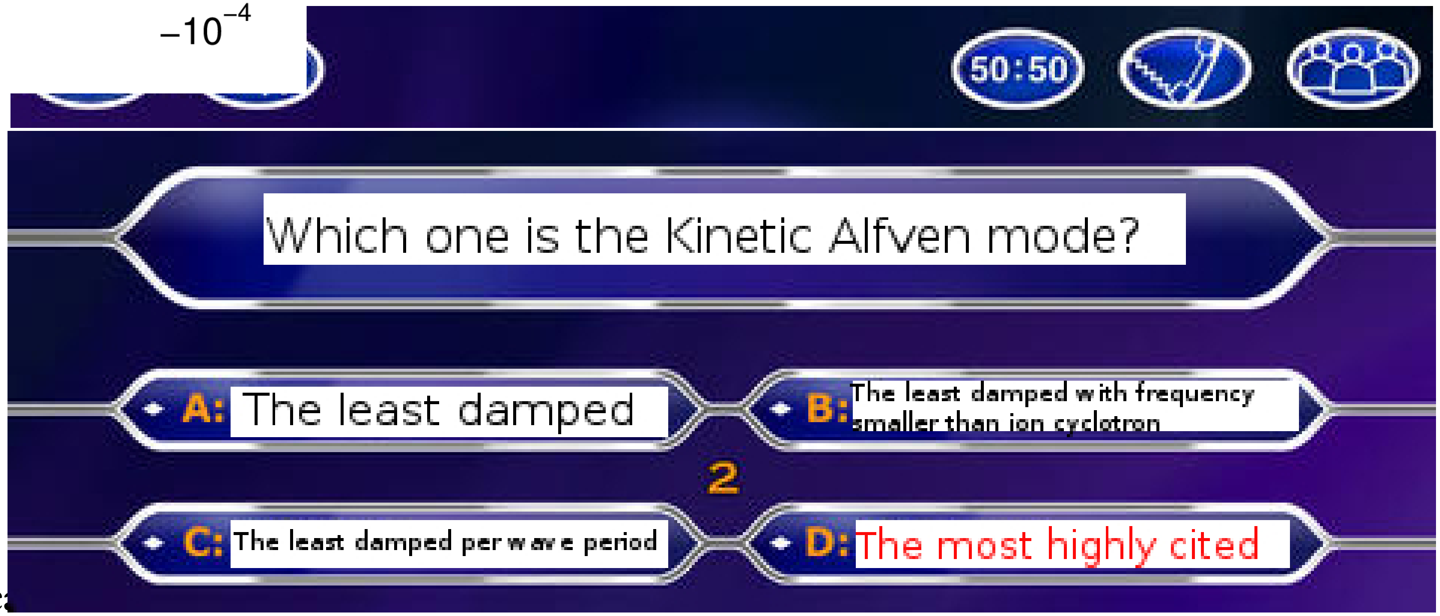
Why are Kinetic Alfven waves so special?



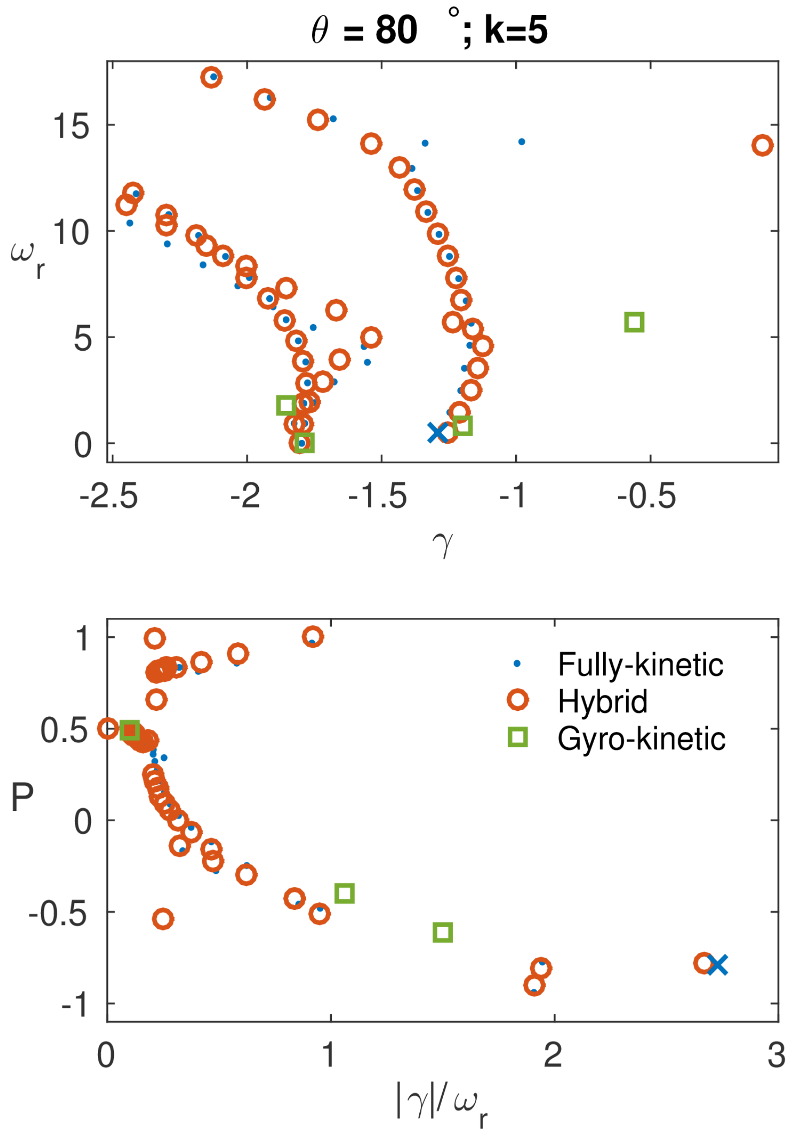


Why are Kinetic Alfven waves so special?





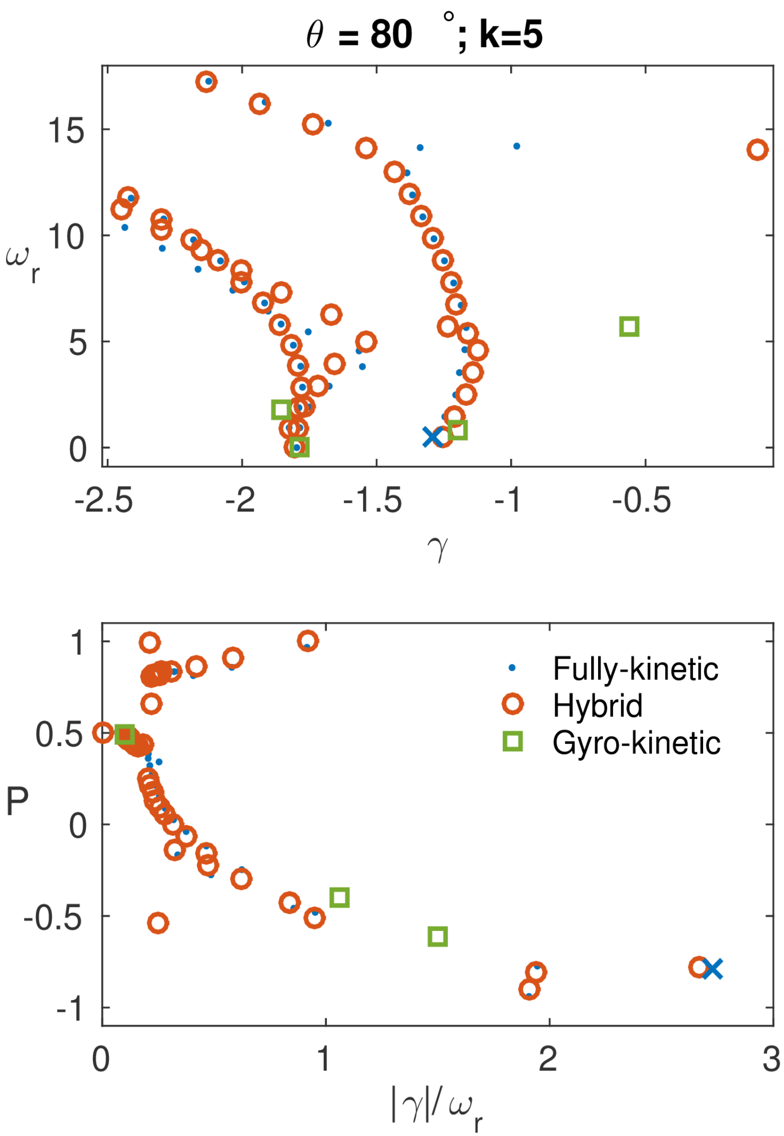
Models comparison



What is a good metric to assess how 'good' a model is?

I believe that one should NOT focus on a single mode, but rather look for a measure that would represent the entire spectrum

Models comparison



What is a good metric to assess how 'good' a model is?

I believe that one should NOT focus on a single mode, but rather look for a measure that would represent the entire spectrum:

The PSEUDOSPECTRUM !!

Ref: Trefethen et al. *Science* (1993)
Trefethen & Embree "Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators" *Princeton Univ. Press* (2005)

E. Camporeale, D. Burgess, T. Passot, *Phys. Plasmas* (2009)
E. Camporeale, T. Passot, D. Burgess, *Astrophys. J.* (2010)
E. Camporeale *Space Sci. Rev.* (2012)

Error based on pseudospectrum

- In all of the three models, the spectrum is numerically calculated by searching for the roots of

$$f(\omega, \mathbf{k}) = \det(\mathbf{D})$$

  with D a 3x3 complex matrix.

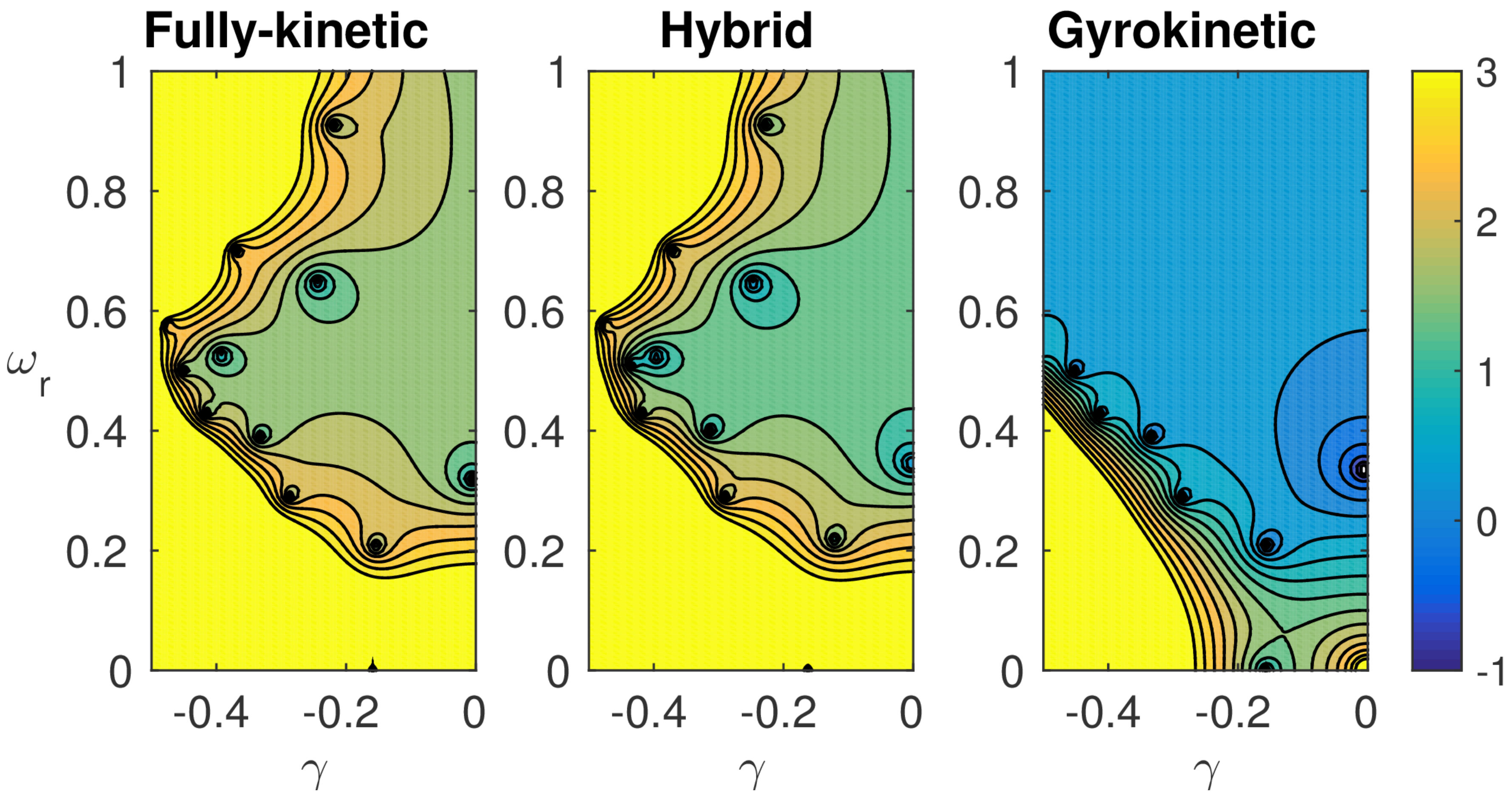
- The pseudo-spectrum can be defined as the region in the complex plane such that

$$\Lambda_\varepsilon(\mathbf{D}) = \{z \in \mathbb{C} : |\det(\boldsymbol{D}(z))| \leqslant \varepsilon\}.$$

- The iso-contours define the displacement of a normal mode due to a perturbation (of magnitude proportional to ε) of the linear operator
- They also show the degree of coupling between normal modes (VM normal modes are non-orthogonal!)

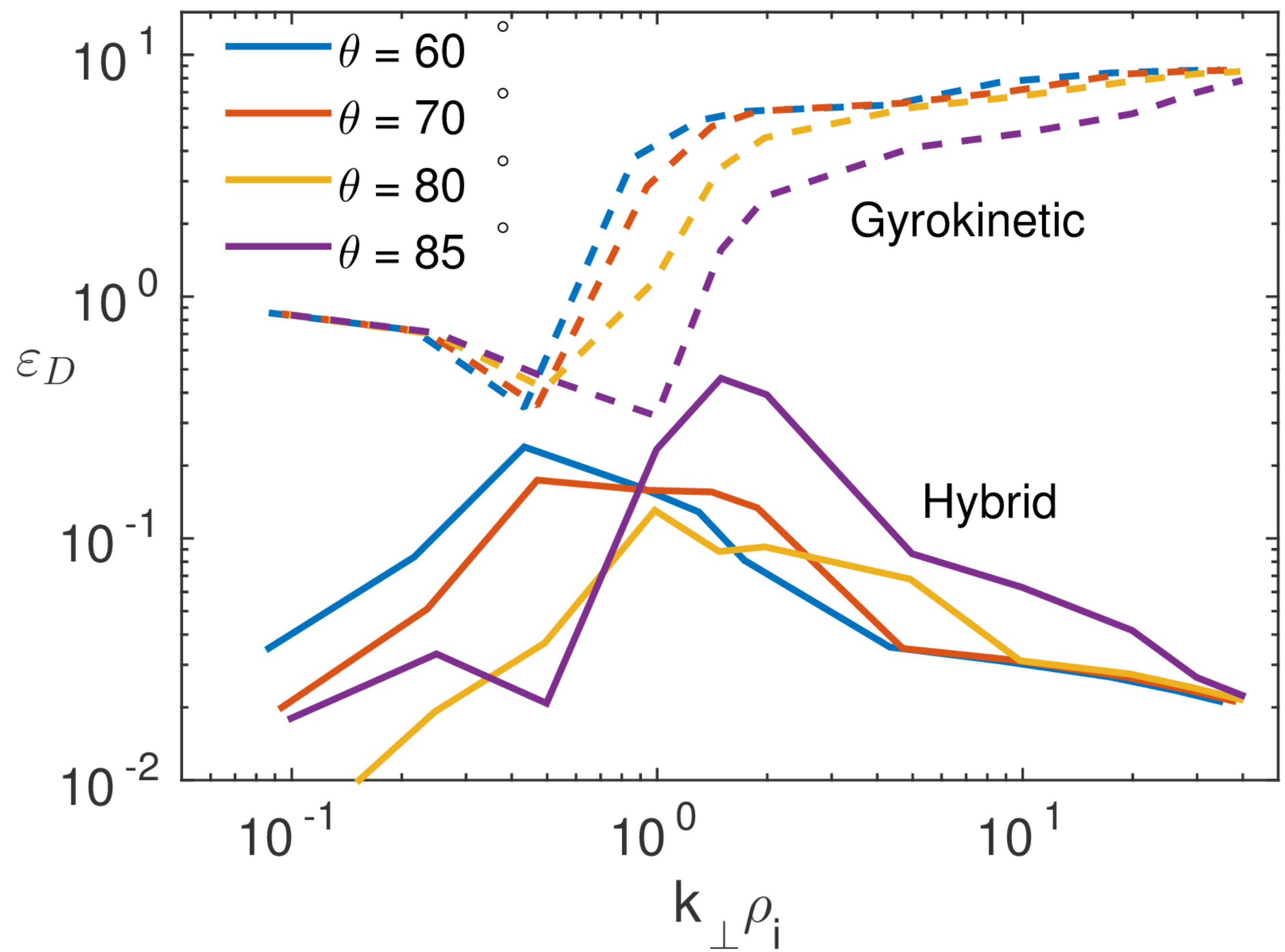
Pseudospectrum comparison

$k\rho_i = 1$; $\theta = 80$ deg



Model error

$$f(\omega, \mathbf{k}) = \det(\mathbf{D}) \qquad \phi(\omega) = 1 + \frac{9f}{1+f}. \qquad \varepsilon_D = \left\| \frac{(\phi(\omega) - \phi(\omega_{VM}))}{\phi(\omega_{VM})} \right\|_2$$



A new quasi-neutral model

- Can we enforce Quasi-neutrality in a fully-kinetic model?
- This would allow to overstep the plasma frequency and the Debye length (like in implicit methods)
- High frequency modes would be factored out from the model rather than numerically damped

PHYSICS OF PLASMAS **22**, 020704 (2015)

Neutral Vlasov kinetic theory of magnetized plasmas

Cesare Tronci[1,a)] and Enrico Camporeale[2,b)]
[1]Department of Mathematics, University of Surrey, Guildford GU2 7XH, United Kingdom
[2]Center for Mathematics and Computer Science (CWI), 1098 XG Amsterdam, The Netherlands

- Vlasov equation for all species s:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = 0\,,$$

- Maxwell equations in the low-frequency limit:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\,, \qquad \mu_0^{-1}\nabla\times\mathbf{B} = \mathbf{J}\,, \qquad \rho = 0\,,$$

$c \to \infty$ (or, equivalently, $\varepsilon_0 \to 0$)

- The electric field is prescribed as:

$$\mathbf{E} = -\boldsymbol{V}_e\times\mathbf{B} + \frac{1}{q_e n_e}\nabla\cdot\mathbb{P}_e + \frac{m_e}{q_e}\left(\frac{\partial \boldsymbol{V}_e}{\partial t} + \boldsymbol{V}_e\cdot\nabla\,\boldsymbol{V}_e\right)$$

where $\boldsymbol{V}_e$ is given by Ampère's law $q_e n_e \boldsymbol{V}_e = \mu_0^{-1}\nabla\times\mathbf{B} - q_i\int \mathbf{v}\, f_i\, d^3\mathbf{v}$

The quasi-neutral Vlasov model

- In practice, for numerical stability, the model is implemented as:

$$\frac{\partial f_s}{\partial t} + \mathrm{v} \cdot \frac{\partial f_s}{\partial \mathrm{x}} + \frac{q_s}{m_s}\Big(\mathrm{E} + \mathrm{v} \times \mathrm{B}\Big) \cdot \frac{\partial f_s}{\partial \mathrm{v}} = 0\,,$$

$$\frac{\partial \mathrm{B}}{\partial t} = -\nabla \times \mathrm{E}\,, \qquad \mu_0^{-1}\nabla \times \mathrm{B} = \mathrm{J}\,, \qquad \rho = 0\,,$$

$$\mu ne\mathbf{E} + \frac{c^2}{4\pi e}\nabla \times (\nabla \times \mathbf{E}) = \frac{1}{m_i}\nabla \cdot \boldsymbol{\Pi} - \frac{1}{m_e}\nabla P_e + \frac{1}{m_e c}(\mathbf{j} \times \mathbf{B}) - \frac{\mu ne}{c}(\mathbf{u}_i \times \mathbf{B}) + \frac{1}{e}\nabla \cdot (\mathbf{u}_i\mathbf{j}) + \frac{1}{e}\nabla \cdot (\mathbf{j}\mathbf{u}_i) - \nabla \cdot \left(\frac{\mathbf{jj}}{ne^2}\right)$$

This is actually the divergence of the pressure tensor

The quasi-neutral Vlasov model

- What if the "particle" current $\mathrm{J}_e = q_e \int \mathrm{v} f_e(\mathrm{x},\mathrm{v})\,\mathrm{dv}$ and the "Ampere's"current $q_e n_e V_e = \mu_0^{-1} \nabla \times \mathrm{B} - q_i n_i V_i$ diverge in time?
- One can show that this definition of electric field

$$\mathrm{E} = -V_e \times \mathrm{B} + \frac{1}{q_e n_e} \nabla \cdot \mathbb{P}_e + \frac{m_e}{q_e} \left(\frac{\partial V_e}{\partial t} + V_e \cdot \nabla V_e \right)$$

where V_e is given by Ampère's law $q_e n_e V_e = \mu_0^{-1} \nabla \times \mathrm{B} - q_i \int \mathrm{v} f_i \, \mathrm{d}^3\mathrm{v}$

ensures that the model is consistent, if the two quantities are equal at initial time;

- Moreover, one can show that the model follows from a variational principle

The quasi-neutral Vlasov model: linear theory test

From Tronci & Camporeale, *Phys. Plasmas* (2015)

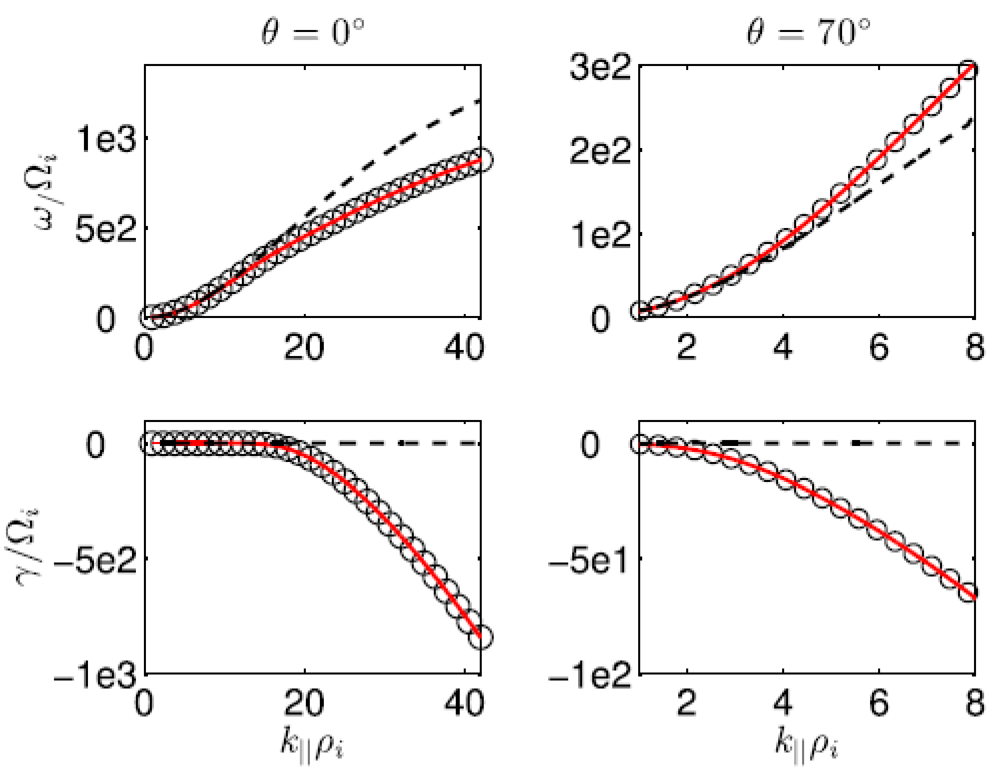


FIG. 1. Real frequency (top) and damping rate (bottom) for Whistler wave propagation at $\theta = 0^\circ$ (left) and $\theta = 70^\circ$ (right). Red line refers to neutral Vlasov, while the dashed line and the circles are used for the hybrid model and Maxwell-Vlasov, respectively.

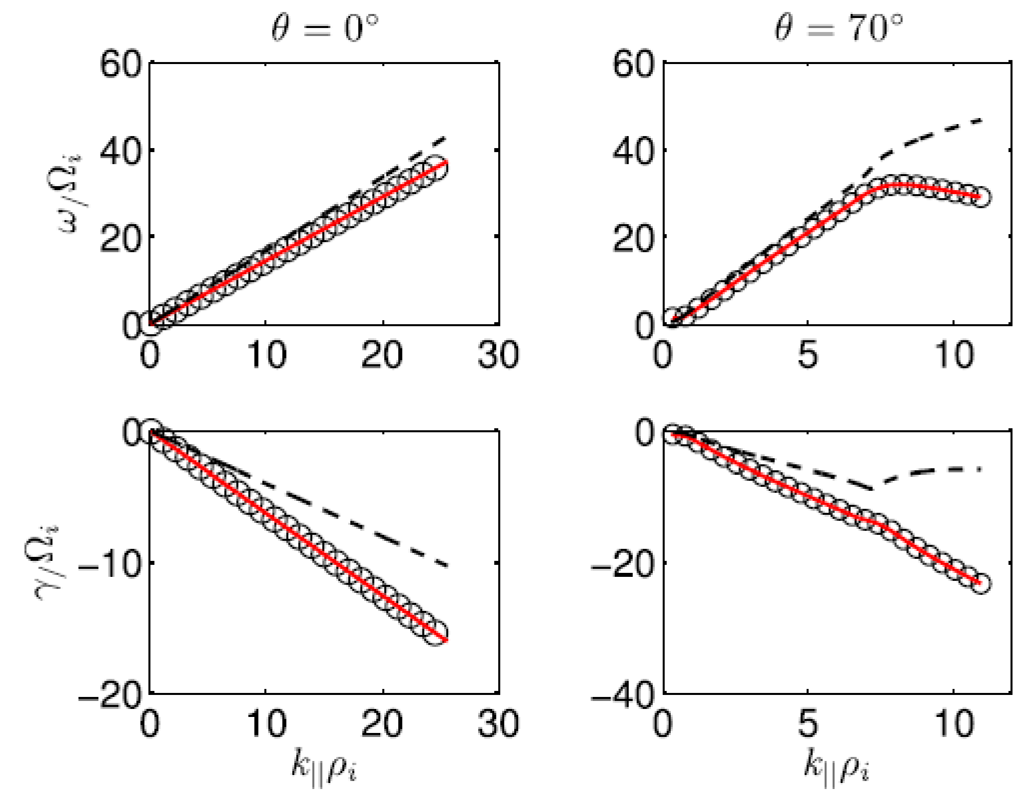


FIG. 2. Real frequency (top) and damping rate (bottom) for Alfvén wave propagation at $\theta = 0^\circ$ (left) and $\theta = 70^\circ$ (right). Legend is as in the previous figure.

Enrico Camporeale – e.camporeale@cwi.nl

Conclusions

We have performed a linear theory comparison between hybrid and gyrokinetic models

- There are no obvious reasons to focus on a single mode, at large propagation angles
- The pseudospectrum allows a more comprehensive analysis of the validity of a model, in the linear regime

Ref: Camporeale & Burgess, *under review*

Conclusions

We have proposed a fully-kinetic quasi-neutral Vlasov model

- This model recovers, as special cases, all quasi-neutral plasma models appeared in the literature (Cheng & Johnson (1999), Hesse & Winske (1993), Valentini et al. (2007) ,Park et al. (1999), etc.)
- Linear theory results are indistinguishable from Vlasov-Maxwell, as long as one does not approach plasma frequency / Debye length
- Non-linear implementation is in progress

Ref: Tronci & Camporeale, *Phys. Plasmas* (2015)