Partial differential equations

- Many problems in science involve the **evolution of quantities** not only **in time** but also **in space** (this is the most common situation)!
- We will call **partial differential equation** a relation involving derivatives with respect both time and spatial coordinates.
- Examples:
  - Poisson's equation for the electrostatic potential

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\rho(x,y,z)}{\epsilon}$$

Partial differential equations

- Fourier's law for heat propagation

$$\frac{\partial T(x,y,z,t)}{\partial t} = \kappa \nabla^2 T(x,y,z) = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

- D'Alembert equation for wave propagation

$$\Box^2 f(x,y,z,t) = \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = 0$$

- When dealing with PDEs one has to know:
- An initial condition, to start the time integration, when present (e.g. in the second and third cases);

Partial differential equations

This means to know the solution (**initial condition**), that is a function of the spatial coordinates at $t=0$:

$$T_0(x, y, z) = T(x, y, z, t = 0)$$

for instance, for Fourier's law.

- We suppose that the spatial coordinates change in a domain: $(x, y, z) \in \mathbb{R}^3$.

  In this case we need to know how the solution (or its derivatives) behaves at the boundaries of the domain.

Partial differential equations

- Generally, the equations that are (mostly) physically meaningful fall into three categories:
  - **Elliptic equations**: a typical example is given by **Poisson's equation** for the electrostatic potential. They typically represent problems in which a field depends on some spatial distribution of sources.
  - **Parabolic equations**: a typical example is given by **Fourier's law** for the transfer of the heat. They represent problems involving diffusion of a field in space.

Partial differential equations

- **Hyperbolic equations**: a typical example is given by **d'Alembert's equation** for the propagation of waves. They typically represent problems describing the propagation of some quantities (signals, waves, etc.).
- Other kind of equations are of course possible, however the three categories summarize the majority of cases which are met typically in science.
- The different physical meaning of the three kind of equations involves quite different approaches and difficulties in finding the solutions!
- Let us examine the three cases separately...

Elliptic differential equations

- Although they do not depend explicitly on time, and therefore require less knowledge (no initial conditions needed!) to solve the problem, **the solution of elliptic problems is by far the most difficult in the general case**!
- In **one dimension** (fields depending only on one single coordinate, e.g. x!) the solution is equivalent to solve a **second order BVP**, like the one we have already solved, thus we have no problem at all!

Elliptic differential equations

- For instance, let us consider the **Poisson's equation** for the **electrostatic potential produced by a spherical distribution of charges**.
- We suppose to have, for instance, a **spherical distribution of charges** $\rho(r)$, where the charge density **depends on the radius**, but not on the other spherical coordinates (the azimuthal coordinate ϕ and the polar coordinate θ).
- The Poisson's equation then becomes:

Elliptic differential equations

$$\nabla^2 V(r) = -\frac{\rho(r)}{\epsilon} \quad \Rightarrow \quad \frac{d^2V(r)}{dr^2} + \frac{2}{r}\frac{dV(r)}{dr} = -\frac{\rho(r)}{\epsilon}$$

with boundary conditions at $r=0$ and $r=R$, where R is, for instance, the radius of the sphere.

- This is, in fact, a second order BVP problem like the one we already solved, with:

$$x = r; \quad Y = V; \quad d(r) = 1;$$

$$f(r) = \frac{2}{r}; \quad g(r) = 0; \quad h(r) = -\frac{\rho(r)}{\epsilon}$$

Elliptic differential equations

- Already in two dimensions, the problem becomes very complicated unless one cannot do some special hypotheses, for instance a periodic direction!
- A typical 2D case in which there is a periodic direction is the Poisson's equation for a distribution of charges in polar coordinates:

$$\nabla^2 V(r,\theta) = -\frac{\rho(r,\theta)}{\epsilon} \quad \Rightarrow$$

$$\frac{\partial^2 V(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial V(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V(r,\theta)}{\partial \theta^2} = -\frac{\rho(r,\theta)}{\epsilon}$$

Elliptic differential equations

- Since the q direction is a periodicity direction, we can write Fourier expansions for the fields depending on this variable, in the form:

$$V(r,\theta) = \sum_{n=-\infty}^{+\infty} \hat{V}_n(r) e^{in\theta}; \qquad \rho(r,\theta) = \sum_{n=-\infty}^{+\infty} \hat{\rho}_n(r) e^{in\theta}$$

- By substituting this expansion in the equation:

$$\frac{\partial^2}{\partial r^2}\left[\sum_{n=-\infty}^{+\infty} \hat{V}_n(r) e^{in\theta}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[\sum_{n=-\infty}^{+\infty} \hat{V}_n(r) e^{in\theta}\right] +$$

$$+ \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left[\sum_{n=-\infty}^{+\infty} \hat{V}_n(r) e^{in\theta}\right] = -\frac{1}{\epsilon}\left[\sum_{n=-\infty}^{+\infty} \hat{\rho}_n(r) e^{in\theta}\right]$$

Elliptic differential equations

- The derivative is a linear operator, therefore we can bring it inside the sums:

$$\sum_{n=-\infty}^{+\infty}\left[\frac{d^2\hat{V}_n(r)}{dr^2}+\frac{1}{r}\frac{d\hat{V}_n(r)}{dr}-\frac{n^2}{r^2}\hat{V}_n(r)\right]e^{i\theta}=-\sum_{n=-\infty}^{+\infty}\frac{\hat{\rho}_n(r)}{\epsilon}e^{i\theta}$$

$$\frac{d^2\hat{V}_n(r)}{dr^2}+\frac{1}{r}\frac{d\hat{V}_n(r)}{dr}-\frac{n^2}{r^2}\hat{V}_n(r)=-\frac{\hat{\rho}_n(r)}{\epsilon}\quad\text{for } n=-\infty,\ldots,+\infty$$

that is we get, **for each value of n**, a **boundary value problem** like the one we have already solved. Of course, by sampling the range of θ–s on N_θ points, n ranges from $-N_\theta/2$ to $+N_\theta/2$!

Elliptic differential equations

- In the general case, for instance in 2D, when no periodicity hypothesis is possible, one can still use the finite differences approximations for the derivatives along each direction to get a relation among the values of the unknown around the points where the derivative is to be computed. For instance, for a Cartesian general 2D case:

$$\left.\frac{\partial^2 V(x,y)}{\partial x^2}\right|_{x_j,y_k} + \left.\frac{\partial^2 V(x,y)}{\partial y^2}\right|_{x_j,y_k} = -\frac{\rho(x_j,y_k)}{\epsilon}$$

where x_j and y_k are descrete points in the domain: $[a,b]$ x $[c,d]$.

Elliptic differential equations

- By using the well-known second-order approximations for the second derivatives:

$$\frac{V(x_{j+1},y_k)+V(x_{j-1},y_k)-2V(x_j,y_k)}{h_x^2}+$$

$$+\frac{V(x_j,y_{k+1})+V(x_j,y_{k-1})-2V(x_j,y_k)}{h_y^2}=-\frac{\rho(x_j,y_k)}{\epsilon}$$

- If we sample the domain [a,b] x [c,d] into N_x intervals along the x direction and N_y along the y direction, that is:

$$j=0,\ldots,N_x; \qquad k=0,\ldots,N_y$$

Elliptic differential equations

- We can use a “linear” index l to represent the single value of the potential V on the generic (x_j, y_k) point of the computational grid:

$$V_l = V(x_j, y_k); \qquad l = j + k \cdot N_x; \qquad l = 0, \ldots, N_x \cdot N_y$$

- In this description, the equation becomes:

$$\frac{1}{h_x^2} V_{l+1} + \frac{1}{h_x^2} V_{l-1} + \frac{1}{h_y^2} V_{l+N_x} + \frac{1}{h_y^2} V_{l-N_x} -$$

$$\left(\frac{2}{h_x^2} + \frac{1}{h_y^2} \right) V_l = -\frac{\rho_l}{\epsilon}$$

Elliptic differential equations

- Therefore, we have got a **sparse system of linear equations** in the N_x x N_y+1 unknown quantities V_l. For large values of N_x and N_y, this is a **huge system of equations**!
- However, we can take advantage of the fact that system has a **sparse matrix** (only 5 terms for each rows of the matrix are different from zero!) to solve it in a “more or less” efficient way with a **relaxation procedure**, like Gauss-Seidel or the Jacobi method.

Parabolic differential equations

- In order to build up a model for a **parabolic equation**, let us suppose we have to solve the following 1D **diffusion equation**:

$$\frac{\partial f(x,t)}{\partial t} = \nu \frac{\partial^2 f(x,t)}{\partial x^2}$$

where:

- ν is a constant;
- x belongs to an interval $[a,b]$;
- we suppose to have some information on the behavior of f (or its derivative) on the boundaries.

Parabolic differential equations

- We suppose to make a **discretization of both time and space**. In particular, we suppose to have in integration interval in time:

$$t \in [0, T_{\text{end}}]; \qquad t_n = nh; \qquad h = T_{\text{end}}/N_t$$

  where N_t is the number of **time subintervals**.

- Analogously, for the spatial coordinate x:

$$x \in [a, b]; \qquad x_j = a + j\Delta x; \qquad \Delta x = \frac{b-a}{N}$$

  where N is the number of **space subintervals**.

Parabolic differential equations

- Let us indicate with an apex the temporal step and with a subscript the spatial subinterval:

$$f(x_j, t_n) = f_j^n; \qquad j = 0, \ldots, N; \qquad n = 0, \ldots, N_t$$

- In a finite difference approach, we can use for instance an explicit scheme for the time discretization (Forward Euler or second-order Runge-Kutta, for instance) and afterwards a centered scheme for the spatial derivatives:

$$\left.\frac{\partial f(x_j, t)}{\partial t}\right|_{t_n} = \nu \left.\frac{\partial^2 f(x, t)}{\partial x^2}\right|_{x_j, t_n}$$

Parabolic differential equations

- With a first order Forward Euler scheme, this becomes:

$$f_j^{n+1} = f_j^n + h\nu \left. \frac{\partial^2 f^n(x)}{\partial x^2} \right|_{x_j}$$

  or, for a second order Runge-Kutta scheme:

$$f_j^* = f_j^n + \frac{h}{2}\nu \left. \frac{\partial^2 f^n(x)}{\partial x^2} \right|_{x_j}$$

$$f_j^{n+1} = f_j^n + h\nu \left. \frac{\partial^2 f^*(x)}{\partial x^2} \right|_{x_j}$$

- Then, one can use the centered scheme for the second order spatial derivative:

Parabolic differential equations

$$\left.\frac{\partial^2 f^n}{\partial x^2}\right|_{x_j} = \frac{f^n_{j+1} + f^n_{j-1} - 2f^n_j}{\Delta x^2}$$

$$\left.\frac{\partial^2 f^*}{\partial x^2}\right|_{x_j} = \frac{f^*_{j+1} + f^*_{j-1} - 2f^*_j}{\Delta x^2}$$

for $j=0, \ldots, N$.

- As usual, by using the centered scheme, we have troubles at j=0 and j=N, since we do not know the values f^n_{-1} and f^n_{N+1}.
- These must be obtained from the boundary conditions.

Parabolic differential equations

- In this case, but also for hyperbolic equations, it make sense to consider, other than the case of Dirichelet and Neumann (and, eventually, Robin) boundary conditions, the case of periodic boundaries, namely when:

$$f(a,t) = f(b,t) \quad \Rightarrow \quad \begin{cases} f_{-1} = f_{N-1} & \text{at } j = 0 \\ f_{N+1} = f_1 & \text{at } j = N \end{cases}$$

- Then, the guidelines to include the boundary conditions in the numerical solutions are the following:

Parabolic differential equations

- For **Dirichlet** b.c.:

  we already know the solution at $j=0$ and $j=N$, therefore we can just put this value in the solution and avoid calculating the derivative in those points at all!

- For **Neumann** b.c.:

  if we know the quantities f'_0 and f'_N, we can use the centered scheme for the first derivative to obtain the values of f_{-1} and f_{N+1} to use in the second derivative at $j=0$ and $j=N$:

Parabolic differential equations

$$f_0'^n = \frac{f_1^n - f_{-1}^n}{2\Delta x} \quad \text{and} \quad f_N'^n = \frac{f_{N+1}^n - f_{N-1}^n}{2\Delta x}$$

$$f_{-1}^n = f_1^n - 2\Delta x f_0'^n \quad \text{and} \quad f_{N+1}^n = f_{N-1}^n + 2\Delta x f_N'^n$$

$$\left.\frac{\partial^2 f^n}{\partial x^2}\right|_{x_0} = \frac{f_1^n + f_{-1}^n - 2f_0^n}{\Delta x^2} = \frac{2(f_1^n - \Delta x f_0'^n - f_0^n)}{\Delta x^2}$$

$$\left.\frac{\partial^2 f^n}{\partial x^2}\right|_{x_N} = \frac{f_{N+1}^n + f_{N-1}^n - 2f_N^n}{\Delta x^2} = \frac{2(f_{N-1}^n + \Delta x f_N'^n - f_N^n)}{\Delta x^2}$$

- Finally, for periodic boundaries we use the cyclic relations seen above to directly write the second derivatives at $j=0$ and $j=N$:

Parabolic differential equations

$$\left.\frac{\partial^2 f^n}{\partial x^2}\right|_{x_0} = \frac{f_1^n + f_{-1}^n - 2f_0^n}{\Delta x^2} = \frac{f_1^n + f_{N-1}^n - 2f_0^n}{\Delta x^2}$$

$$\left.\frac{\partial^2 f^n}{\partial x^2}\right|_{x_N} = \frac{f_{N+1}^n + f_{N-1}^n - 2f_N^n}{\Delta x^2} = \frac{f_1^n + f_{N-1}^n - 2f_N^n}{\Delta x^2}$$

- Hence, by starting from the initial condition:

$$f(x_j, t = 0) = f_0(x_j)$$

that is f_j^0 we can compute the second derivative for any value of j and apply the time scheme to get the solution for $n > 0$!

Parabolic differential equations

- To ensure that the method is **numerically correct**, we must take care of eventual **stability problems**.
- To examine stability by using the Von Neumann **stability criterion**, we need an analytic tool to compute easily the spatial derivatives. This is possible actually only for periodic functions, by supposing that our solution has the form of a single Fourier mode, like: $f(x_j, t_n) = A(t_n)e^{ikx_j}$
- Namely, we are supposing the signal is a single wave of amplitude A changing in time.

Parabolic differential equations

- Let us apply this idea to the Forward Euler scheme, which is the simplest one (we know that RK has more or less the same stability!):

$$f_j^{n+1} = f_j^n + h\nu \frac{f_{j+1}^n + f_{j-1}^n - 2f_j^n}{\Delta x^2}$$

$$A(t_{n+1})e^{ikx_j} = A(t_n)e^{ikx_j} + h\nu \frac{A(t_n)e^{ikx_{j+1}} + A(t_n)e^{ikx_{j-1}} - 2A(t_n)e^{ikx_j}}{\Delta x^2} =$$

$$= A(t_n)\left[e^{ikx_j} + h\nu \frac{e^{ik(x_j+\Delta x)} + e^{ik(x_j-\Delta x)} - 2e^{ikx_j}}{\Delta x^2}\right] =$$

$$= A(t_n)e^{ikx_j}\left[1 + h\nu \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2}\right] =$$

$$= A(t_n)e^{ikx_j}\left[1 + 2h\nu \frac{\cos(k\Delta x) - 1}{\Delta x^2}\right] =$$

Parabolic differential equations

- The Von Neumann's stability criterion then becomes:

$$\left|\frac{f_j^{n+1}}{f_j^n}\right| \leq 1 \quad \Rightarrow \quad \left|\frac{A(t_{n+1})}{A(t_n)}\right| \leq 1$$

$$\left|1 + 2h\nu\frac{\cos(k\Delta x) - 1}{\Delta x^2}\right| \leq 1 \quad \Rightarrow \quad \begin{cases} -1 \leq 1 + 2h\nu\frac{\cos(k\Delta x)-1}{\Delta x^2} \\ 1 + 2h\nu\frac{\cos(k\Delta x)-1}{\Delta x^2} \leq 1 \end{cases}$$

- The second inequality is always satisfied, since:

$$\cos(k\Delta x) - 1 \leq 0$$

and all other quantities are positive, by definition!

Parabolic differential equations

- The first one gives, as a result:

$$\frac{h\nu[1-\cos(k\Delta x)]}{\Delta x^2} \leq 1$$

  that is always satisfied if $\frac{h\nu}{\Delta x^2} \leq 1 \Rightarrow h \leq \frac{\Delta x^2}{\nu}$

- In order to understand better this condition (but also to check the correctness of the numerical solution!), it is useful to see what the **analytical solution** of the original equation is, at least in some particular case.

Parabolic differential equations

- The **analytical solution** can be “easily” found only in the **periodic case**, in which we can suppose that the solution in the x direction can be expanded as a Fourier series (for simplicity, we suppose that the periodicity interval is equal to 2π!):

$$f(x,t) = \sum_{k=-\infty}^{+\infty} \hat{f}_k(t) e^{ikx}$$

- Under these hypotheses, we can write:

$$\frac{\partial f(x,t)}{\partial t} = \sum_{k=-\infty}^{+\infty} \frac{d\hat{f}_k(t)}{dt} e^{ikx}$$

Parabolic differential equations

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \sum_{k=-\infty}^{+\infty} \hat{f}_k(t) \frac{d^2}{dx^2} e^{ikx} = \sum_{k=-\infty}^{+\infty} -k^2 \hat{f}_k(t) e^{ikx}$$

- These two derivatives must be equal, according to the equation, therefore:

$$\frac{d\hat{f}_k(t)}{dt} = -\nu k^2 \hat{f}_k(t) \quad \Rightarrow \quad \hat{f}_k(t) = \hat{f}_k(t=0) e^{-\nu k^2 t}$$

- The final form of the solution is, therefore:

$$f(x,t) = \sum_{k=-\infty}^{+\infty} \hat{f}_k(0) e^{-\nu k^2 t} e^{ikx}$$

Parabolic differential equations

- This means that in the original signal, each Fourier component remains spatially unaltered, but it is dissipated in time for a factor:

$$e^{-\nu k^2 t} = e^{-t/\tau_k}$$

- The quantity τ_k is then the **characteristic time of dissipation** for the k-th Fourier harmonic in the signal. Since: $\tau_k = 1/(\nu\, k^2)$, it is shorter for higher values of k. On the other hand, high k-s mean quick oscillations...

Parabolic differential equations

- Therefore, if the signal is a **superposition of several spatial signals**, some varying slowly and some changing more rapidly, all of them are exponentially damped, but **the latter are damped much faster than the former**!
- We can think of this phenomenon as a "**smoothing**" of the signal, the rapid changes present in the signal being damped faster than the slow variations.
- Since $k_{MAX} = N/2$, $\tau_{kMAX} = \Delta x^2/(\pi^2 \nu)$, the Von Neumann's criterion then affirms that: $h < \pi^2 \tau_{kMAX}$

Hyperbolic differential equations

- As we said, hyperbolic equations describe propagation of signals.
- A typical hyperbolic equation is:

$$\frac{\partial f}{\partial t} \pm c\frac{\partial f}{\partial x} = 0$$

- The d'Alembert equation cited above can be written as two equations of this kind:

$$\Box^2 f = \frac{\partial^2 f}{\partial t^2} - v^2\frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f = 0$$

Hyperbolic differential equations

- It is possible to show that the solution of such equations is in the form:

$$f(x,t) = f(x \mp ct)$$

that is **any functional form is a solution**, if the dependency on x and t is in the form of a function:

$$\xi(x,t) = x \mp ct$$

- In fact, we can show immediately that this is a solution of the above equation, by substituting the given solution and showing that we get an identity:

Hyperbolic differential equations

- In fact, we have:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} f[\xi(x,t)] = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} = \mp c\frac{df}{d\xi}$$
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f[\xi(x,t)] = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = \frac{df}{d\xi}$$

- Then, substituting these quantities into the equation, we get:

$$\mp c\frac{df}{d\xi} \pm c\frac{df}{d\xi} = 0$$

that confirms the hypothesis.

Hyperbolic differential equations

- What is the meaning of such a solution?
- Let us consider first the solution: $f(x\text{-}ct)$.
- If we consider the plane x-t and the lines on which the quantity:

$$x - ct = k$$

remains constant, these form, when varying k, a **bundle of parallel lines**, with **positive angular coefficient** ($1/c$), having equations (in the x-t plane):

$$t = \frac{x}{c} - \frac{k}{c}$$

Hyperbolic differential equations

- Along these lines, called characteristic curves of the equation, the solution remains constant.
- This means that, if at $t=0$ I have a **generic profile** which represent the solution of the equation, this profile will **remain unchanged along the characteristic lines**.
- This means that the solution is "**transported**" along the characteristics, that means **it is moving along the positive part of the x axis**, its profile remaining unaltered during the transport.

Hyperbolic differential equations

- In the same way, a solution of the form: $f(x+ct)$, will remain constant along the characteristic lines of equation:

$$t = -\frac{x}{c} + \frac{k}{c}$$

that have a **negative angular coefficient**, that is equivalent to a **solution transported along the negative part of the x axis**.

- Summarizing, a solution: $f(x-ct)$ corresponds to **signal propagating to the right**, $f(x+ct)$ to a **signal propagating to the left**.

Hyperbolic differential equations

- How do we solve these equations numerically?
- At first, the solution seems to be **elementary**!
- We can use, for instance, an explicit time scheme, as we did for parabolic equations, for time advancement, and a centered scheme for the spatial derivatives.
- Namely, something like:

$$\left.\frac{\partial f}{\partial t}\right|_{t_n,x_j} = -c \left.\frac{\partial f}{\partial x}\right|_{t_n,x_j}$$

Hyperbolic differential equations

- Then, we can for instance apply a Forward-Euler scheme for approximating the time derivative:

$$f_j^{n+1} = f_j^n - ch \left. \frac{\partial f}{\partial x} \right|_{x_j}$$

$$f_j^{n+1} = f_j^n - ch \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

or, the analogous formulas for a second order Runge-Kutta scheme:

$$f_j^* = f_j^n - c\frac{h}{2} \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

$$f_j^{n+1} = f_j^n - ch \frac{f_{j+1}^* - f_{j-1}^*}{2\Delta x}$$

Hyperbolic differential equations

- Too bad that **it doesn't work**!
- Indeed, we can show that such a scheme is **always unstable**, for any value of h!
- To carry out the stability analysis, as usual we use the Forward-Euler scheme, which is simpler. Again, we consider a solution in the form:

$$f_j^n = A(t_n)e^{ikx_j}$$

- The Von Neumann's criterion, applied to the equation becomes:

$$A(t_{n+1})e^{ikx_j} = A(t_n)e^{ikx_j} - ch\frac{A(t_n)e^{ikx_{j+1}} - A(t_n)e^{ikx_{j-1}}}{2\Delta x}$$

Hyperbolic differential equations

- Since: $x_{j\pm1} = x_j \pm \Delta x$ this relation becomes:

$$A(t_{n+1})e^{ikx_j} = A(t_n)e^{ikx_j} - chA(t_n)\frac{e^{ik(x_j+\Delta x)} - e^{ik(x_j-\Delta x)}}{2\Delta x} =$$

$$A(t_n)e^{ikx_j} - chA(t_n)e^{ikx_j}\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} =$$

$$A(t_n)e^{ikx_j}\left(1 - chi\frac{\sin k\Delta x}{\Delta x}\right)$$

- The Von Neumann stability condition gives:

$$\left|\frac{f_j^{n+1}}{f_j^n}\right| \leq 1 \quad \Rightarrow \left|1 + chi\frac{\sin(k\Delta x)}{\Delta x}\right| \leq 1$$

Hyperbolic differential equations

- In this case, the amplification factor is complex, therefore we have to consider the complex modulus of the quantity:

$$\sqrt{1 + c^2 h^2 \frac{\sin^2(k\Delta x)}{\Delta x^2}} \leq 1$$

  which, of course, is **never satisfied** for any value of h!

- There are **several possible solutions** to this problem!

Hyperbolic differential equations

- The most famous of them is the so-called **Upwind method**.
- The idea behind this method is that, physically, the **hyperbolic equation represents the propagation of a signal** (information) along a well-specified direction (right or left, according to the sign of the speed c!). Therefore, when we use a centered scheme we are assuming that we have already the information necessary to propagate the signal both to the left and to the right but **this is not true**!

Hyperbolic differential equations

- This assumption leads to the instability!
- Therefore, we can recover a **stable scheme** if we simply use the information we have already, that is if we **compute the derivatives** by using the points to the **left of the signal**, when it is propagating **towards the right** ($c > 0$) and, vice versa, the **points to the right**, when it is **propagating towards the left** ($c < 0$).
- In formulas:

$$f_j^{n+1} = f_j^n - c \begin{cases} \frac{f_j^n - f_{j-1}^n}{\Delta x} & \text{if } c > 0 \\ \frac{f_{j+1}^n - f_j^n}{\Delta x} & \text{if } c < 0 \end{cases}$$

Hyperbolic differential equations

- It is possible to show, by repeating the Von Neumann's analysis to thie scheme, that the scheme is stable if:

$$h < \frac{\Delta x}{c}$$

- This is the **Courant-Friedricks-Lewy (CFL)** condition, that expresses the fact that the time step (h) must be smaller than the characteristic time with which the signal propagates on the numerical grid ($\Delta x/c$), in order the scheme to be stable.

Hyperbolic differential equations

- Another possible solution is the **Lax-Wendroff** scheme:

$$f_j^{n+1} = \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - ch\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

  which is nothing else than the Forward-Euler centered scheme, modified by taking the average of the solution between x_{j+1} and x_{j-1} at time t, instead of f_j^n.

- Also in this case, **the scheme is stable** if the CFL condition is satisfied!

Hyperbolic differential equations

- Another possibility is to add to the equation an "**artificial**" **dissipative term**, that is we introduce arbitrarily **a parabolic term in the equation**, which therefore becomes a **hyperbolic-parabolic equation**, to make the centered scheme stable.
- In other words, we solve the equation:

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \nu \frac{\partial^2 f}{\partial x^2}$$

with the idea that, **for vanishing ν, we recover the original equation**.

Hyperbolic differential equations

- The Forward-Euler scheme, in this case, becomes:

$$f_j^{n+1} = f_j^n + h\left[-c\frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x} + \nu\frac{f_{j+1}^n + f_{j-1}^n - 2f_j^n}{\Delta x^2}\right]$$

- This may look as a "**dirty trick**", but in fact it is easy to show that the upwind and Lax-Wendroff schemes both work because **the numerical error** of the spatial derivatives **act as a dissipative term**, which smooths the instabilities.

Hyperbolic differential equations

- For this reason, it is **far better** to introduce an "artificial" dissipation, that can be "**controlled**" by the user, instead of a "numerical" dissipation, whose effects depend on the choice of grid-points, time step, and on the other parameters of the equation, which cannot be "**controlled**" *a priori* by the user.
- It is possible to show that the stability condition becomes, in this case: $h \leq \frac{2\nu}{c^2 + \nu^2 k^2}$ that is indeed vanishing for vanishing ν !

Burgers' equation

- The Burgers' equation is a special case of **non-linear** hyperbolic-parabolic equation:

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = \nu\frac{\partial^2 f}{\partial x^2}$$

- The characteristic of equations is that the **speed** is at each spatial point equal to the **solution itself**.
- This means that, the **positive parts** of the solution **travel to the right** of the domain, whilst the **negative parts to the left**, in such a way that **strong gradients** are produced in the solution (**shock fronts**)!

Burgers' equation

- This equation can be solved both with **dissipative centered schemes** and with **upwind schemes**. In the former case, one has to ensure that the artificial dissipation is able to effectively "smooth" the discontinuity in the solution brought by the non-linear term, that is:

$$f\frac{\partial f}{\partial x} \sim \nu\frac{\partial^2 f}{\partial x^2}$$

$$\frac{f^2}{\Delta x} \sim \nu\frac{f}{\Delta x^2}$$

$$\nu \sim f\Delta x$$