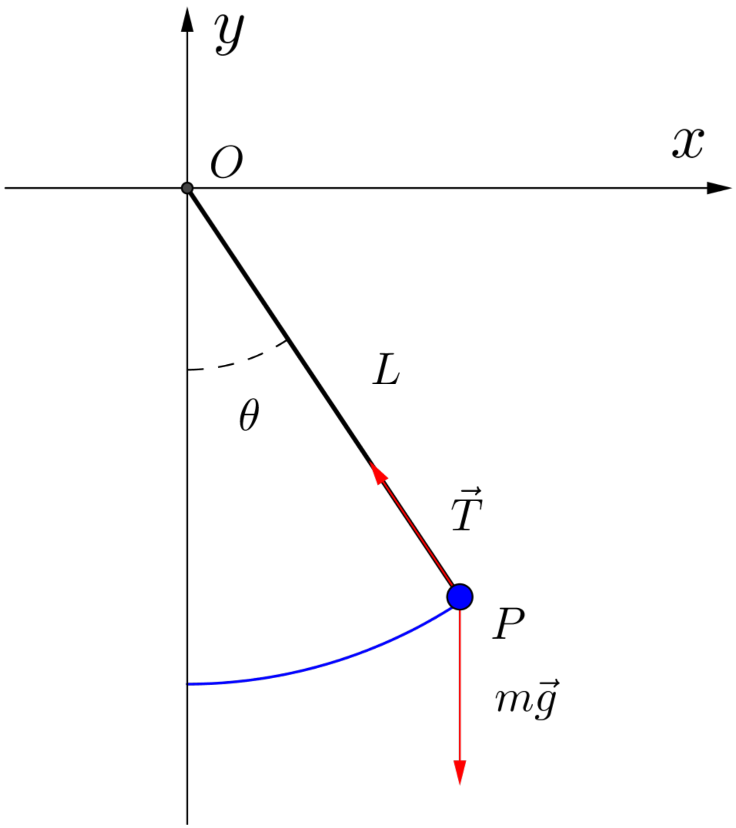
ODE Examples

- We have seen so far that the **second order Runge-Kutta time scheme is a good compromise for solving ODEs** with a good precision, without making the calculations too heavy!
- It is **unstable for oscillators**, that means that if we do not need to integrate the equations for a long time, we can accept that the energy is not actually conserved but increases in time.
- Whenever we need the conservation of energy is well respected, we have to use other schemes (for instance, the symplectic scheme).

Pendulum

- A first example of a non-linear equation is the so-called “simple pendulum”:



$$\mathbf{OP} = L\sin\theta\hat{\mathbf{i}} - L\cos\theta\hat{\mathbf{j}}$$

$$\mathbf{T} = -T\sin\theta\hat{\mathbf{i}} + T\cos\theta\hat{\mathbf{j}}$$

$$m\mathbf{g} = -mg\hat{\mathbf{j}}$$

$$\mathbf{v} = \frac{d\mathbf{OP}}{dt} = L\left(\frac{d\theta}{dt}\right)\cos\theta\hat{\mathbf{i}} + $$

$$+ L\left(\frac{d\theta}{dt}\right)\sin\theta\hat{\mathbf{j}}$$

Pendulum

- The acceleration of the point is:

$$\mathbf{a} = L\left[\left(\frac{d^2\theta}{dt^2}\right)\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right]\hat{\mathbf{i}} + \\ + L\left[\left(\frac{d^2\theta}{dt^2}\right)\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right]\hat{\mathbf{j}}$$

- For the second Newton's law:

$$m\mathbf{a} = m\mathbf{g} + \mathbf{T} = -mg\hat{\mathbf{j}} - T\sin\theta\hat{\mathbf{i}} + T\cos\theta\hat{\mathbf{j}}$$

- By decomposing along the x and y components, we have:

Pendulum

$$\left[\left(\frac{d^2\theta}{dt^2}\right)\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right] = -\frac{T\sin\theta}{mL}$$

$$\left[\left(\frac{d^2\theta}{dt^2}\right)\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right] = \frac{T\cos\theta}{mL} - \frac{g}{L}$$

- If we multiply the first equation for $\cos\theta$ and the second for $\sin\theta$, adding the two and using the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$, we have:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Pendulum

- Since both g and L are positive constants, we can assume that:

$$\omega^2 = \frac{g}{L}$$

- The motion equation is then given by:

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta$$

- First, we can notice that:
  - This is a second order **non-linear** differential equation;
  - For small θ angles, it reduces to the harmonic oscillatory equation (this is why **watches work**!!!)

Pendulum

- This equation is non-linear, so we do not know the analytical solution. However, some indications about the correctness of the solution can be obtained from energy conservation:

$$E = T + U = \frac{1}{2}mv^2 + mgy =$$

$$= \frac{1}{2}m\left[L^2\left(\frac{d\theta}{dt}\right)^2\cos^2\theta + L^2\left(\frac{d\theta}{dt}\right)^2\sin^2\theta\right] +$$

$$+ mg(-L\cos\theta) = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 - mgL\cos\theta$$

Pendulum

- Since E, m, L are all **constant in time**, also the quantity:

$$\frac{2E}{mL^2} = \left(\frac{d\theta}{dt}\right)^2 - \frac{2g}{L}\cos\theta = \left(\frac{d\theta}{dt}\right)^2 - 2\omega^2\cos\theta$$

must be constant. In particular, at any given time t, it must be equal to the same quantity at $t=0$:

$$\left(\frac{d\theta}{dt}\right)^2 - 2\omega^2\cos\theta = \left.\left(\frac{d\theta}{dt}\right)^2\right|_{t=0} - 2\omega^2 \left.\cos\theta\right|_{t=0}$$

Pendulum

- If we suppose to start at $t=0$ with:

$$\theta(t=0)=0$$

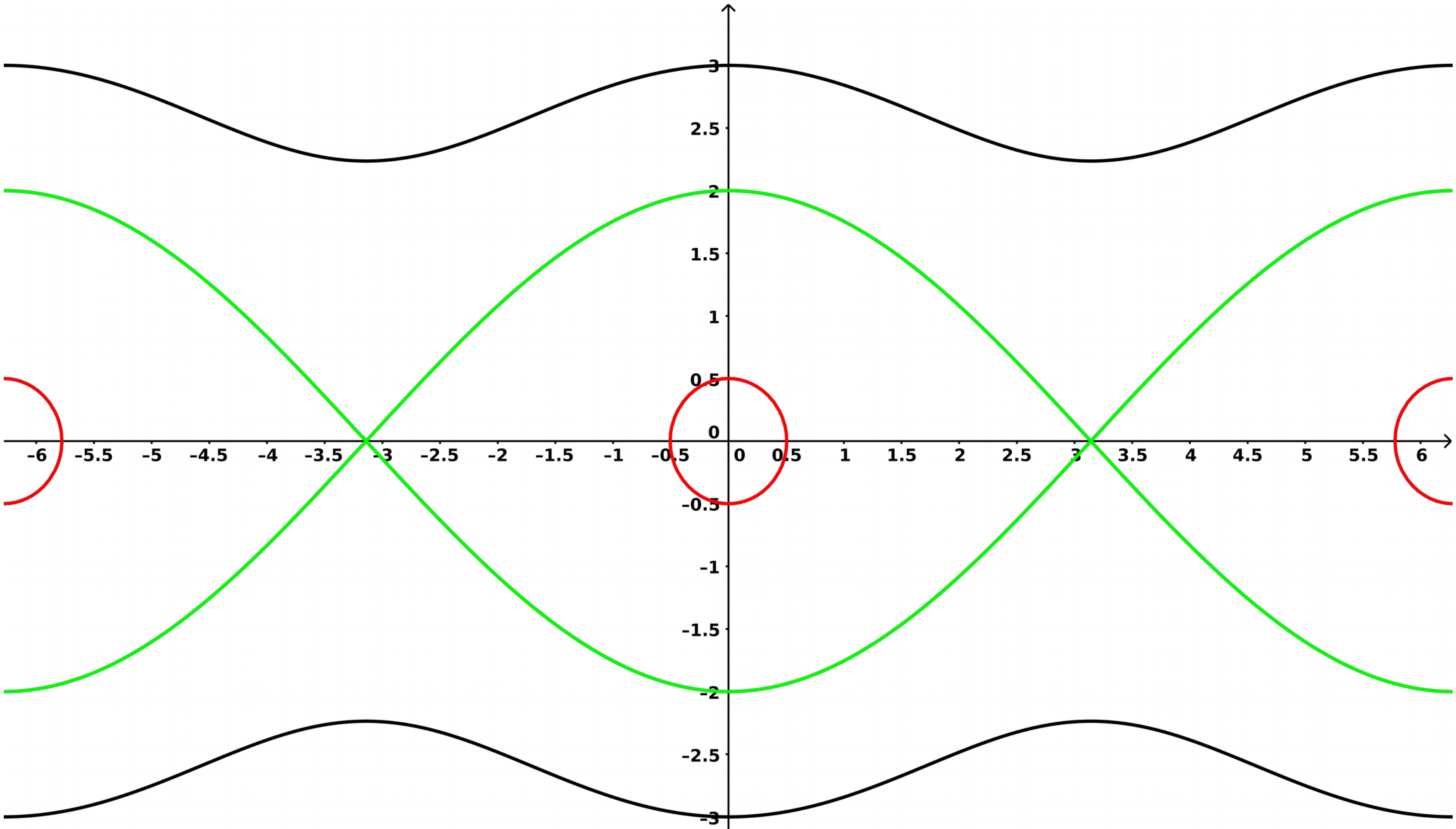
$$\left(\frac{d\theta}{dt}\right)\Big|_{t=0}=\Omega_0$$

we get the following expression for the derivative of θ as a function of θ itself:

$$\left(\frac{d\theta}{dt}\right)=\pm\sqrt{\Omega_0^2-2\omega^2\left(1-\cos\theta\right)}$$

Pendulum

- The plot of this function in the plane: $(d\theta/dt,\ \theta)$, (the so-called **phase space**) for increasing Ω:



Forced and damped oscillator

- A **forced and damped oscillator** is like a harmonic oscillator but we retain the possibility to have friction between the body and the plane and the plane itself is oscillating with an external frequency ω:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = F\cos(\omega t)$$

where:

- γ represents the friction;
- ω_0 is the natural frequency of the oscillator;
- F and ω are the external force (per unit mass) and forcing frequency.

Forced and damped oscillator

- This equation is **linear**, and it is easy to show that the analytical solution of this equation is:

$$x(t) = Ae^{-\gamma t}\sin(\omega_0 t + \phi) + B\cos(\omega_0 t - \epsilon)$$

where B and ε are given by:

$$B = \frac{F}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$

$$\epsilon = \arctan\left(\frac{2\gamma\omega}{\omega^2 - \omega_0^2}\right)$$

- B represents the **long-term amplitude** of the oscillation (in the limit for infinite t!)

Forced and damped oscillator

- It is remarkable that B depends on the external forcing frequency ω. In particular, we have a maximum value of B when the denominator is minimum, that is:

$$\frac{d}{d\omega}\left[(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2\right]=0\Rightarrow$$

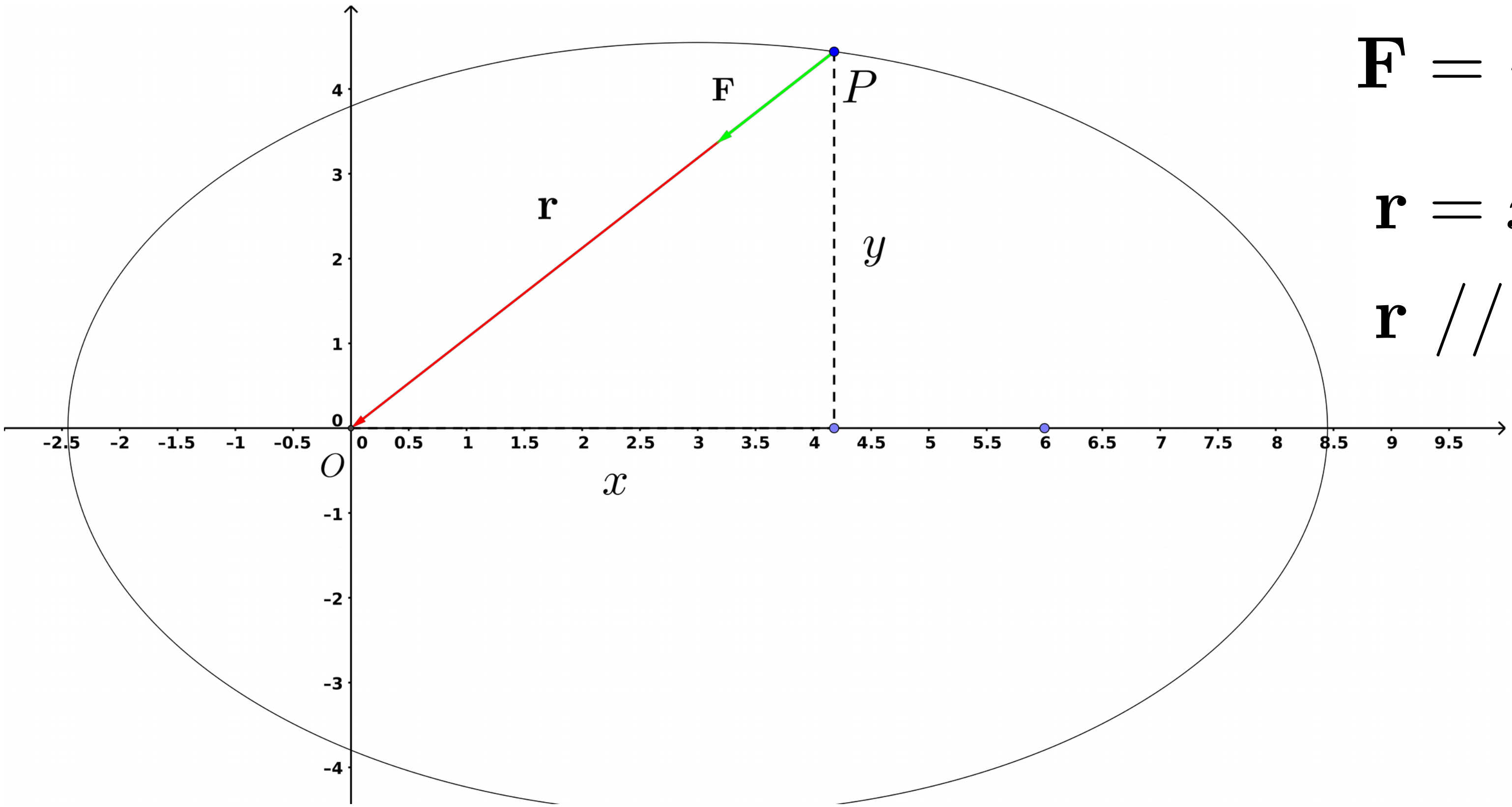
$$\Rightarrow 2(2\omega)(\omega^2-\omega_0^2)+8\gamma^2\omega=0\Rightarrow$$

$$\Rightarrow \omega=\pm\sqrt{\omega_0^2-2\gamma^2}$$

- This means that we have a **resonance** when the external forcing frequency ω takes this value!

Motion of planets

- The force attracting a planet (e.g. the Earth) orbiting around a star (e.g. the Sun) is described by the Newton's gravitational law:



$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r}$$

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

$$\mathbf{r} \;//\; \mathbf{F} \Rightarrow \mathbf{r} \times \mathbf{F} = 0$$

Motion of planets

- From the second law of dynamics:

$$m\mathbf{a} = \mathbf{F}$$

we obtain the system of equations:

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x$$
$$\frac{d^2y}{dt^2} = -\frac{GM}{r^3}y$$

where: $r = \sqrt{x^2 + y^2}$

- This is a **non-linear system of equations** in the unknowns x and y.

Motion of planets

- G = 6.67 x 10^{-11} Nm^2/Kg^2
- M(Sun) = 2 x 10^{30} Kg
- The total energy (per unit mass) must be conserved:

$$\frac{E}{m} = \frac{1}{2}v^2 - \frac{GM}{r}$$

along with the angular momentum:

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}(m\mathbf{r} \times \mathbf{v}) = m\underbrace{\left(\frac{d\mathbf{r}}{dt} \times \mathbf{v}\right)}_{=0} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{r} \times \mathbf{F} = 0$$

Motion of planets

- This gives the second Kepler's law:

$$\mathbf{r} \times \mathbf{v} = \text{constant} = xv_y - yv_x$$

that is also sometimes expressed in an equivalent form as:

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

- Finally Kepler's third law states that:

The square of the orbital period is directly proportional to the cube of the semi-major axis of its orbit.

Lotka-Volterra system

- Let us suppose to have a closed ecosystem made of two species of animals, the first one (preys) being a typical food for the second one (predators).
- If the two species are kept separated one from the other, the preys (if they have enough food!) will grow in number during time. The predators will starve (unless they find an alternative food).
- The Irish economist **Malthus (1766-1834)** predicted that, under such conditions, preys were exponentially increasing in number and predators exponentially decreasing.

Lotka-Volterra system

- Under such conditions (p=preys, P=predators):

$$\frac{dp}{dt} = ap$$
$$\frac{dP}{dt} = -bP$$

- If we now let the two species interact among them, we will observe that the preys decrease with **a rate proportional to the number of predators** and the latter will **increase in number proportionally to the how many preys** are into the system.

Lotka-Volterra system

- This means the equations are modified as:

$$\frac{dp}{dt} = ap - cpP$$
$$\frac{dP}{dt} = -bP + epP$$

where a, b, c, e are constant parameters.

- These is the so-called Lotka-Volterra system of equations, independently developed in 1926 by the Italian mathematician Vito Volterra (1860-1940) and American statistician Alfred J. Lotka (1880-1949).

Lotka-Volterra system

- Volterra demonstrated two laws:

1)The system has cyclic solutions (predators and preys sustain each other);

2)The **averages** of the two populations are **constant in time** and depend on the parameters a, b, c and e:

$$\frac{1}{T}\int_0^T p(t)dt = \frac{b}{e}$$

$$\frac{1}{T}\int_0^T P(t)dt = \frac{a}{c}$$

Lotka-Volterra system

- The latter can be easily demonstrated starting from the system of equations:

$$\frac{1}{p}\frac{dp}{dt} = a - cP$$

$$\frac{1}{P}\frac{dP}{dt} = -b + ep$$

and, by integrating over a period [$0,T$]:

$$\log\frac{p(T)}{p(0)} = aT - c\int_0^T P(t)dt$$

$$\log\frac{P(T)}{P(0)} = -bT + e\int_0^T p(t)dt$$

Lotka-Volterra system

- From the first law, since T is the period:

$$p(T) = p(0)$$
$$P(T) = P(0)$$

from which it follows:

$$0 = aT - c\int_0^T P(t)dt$$
$$0 = -bT + e\int_0^T p(t)dt$$

that is what was to demonstrate.

Lotka-Volterra system

- Remarkably, this simple system was able to explain the behavior of some simple ecosystems. It was able also to explain some more complicated observations concerning biological systems.
- In 1868 in California was introduced an insect (Icerya Purchasi) from Australia, that destroyed all citrus crops. To solve the problem was introduced a ladybug (Novius Cardinalis), a predator for the Icerya.

Lotka-Volterra system

- After the introduction of DDT (Dichloro-diphenyl-trichloro-ethane), it was observed that the number of Iceryas was increasing again.
- This can be explained in terms of Lotka-Volterra's laws, modified to keep into account that the insecticide destroys in the same way both the preys and the predators:

$$\frac{dp}{dt} = ap - cpP - \epsilon p = (a - \epsilon)p - cpP$$
$$\frac{dP}{dt} = -bP + epP - \epsilon P = -(b + \epsilon)P + epP$$

Lotka-Volterra system

- This means, for the second Volterra's law, that the average **number of predators** (Novius Cardinalis) **has decreased**, becoming:

$$\frac{a - \epsilon}{c}$$

  whilst the **number of preys** (Icerya Purchasi) **has increased** to:

$$\frac{b + \epsilon}{e}$$

- This was indeed observed experimentally and was one of the big successes of Lotka-Volterra systems.